SPECTRA ON GENERALIZED DUNFORD INTEGRAL

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To My Family

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CHAPTER I

INTRODUCTION

A function $f(T_1,T_2,\ldots T_N)$ of operators $T_1,T_2,\ldots,T_N$ was introduced by Dr. Von Heyn.† This operator is defined by the following integral:

$$f(T_1,T_2,\ldots,T_N) = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\zeta_1,\zeta_2,\ldots,\zeta_N)\cdot$$

$$\cdot R_{\zeta_1}(T_1)R_{\zeta_2}(T_2)\ldots R_{\zeta_N}(T_N)\cdot d\zeta_1\cdot d\zeta_2\ldots d\zeta_N,$$

where $T_k \in B(X,X)$ is a bounded linear operator from Banach space X into X for each k, and C_k's are simple closed oriented rectifiable curves and are boundaries of U_k's, $U_k \supset \sigma(T_k)$ $k = 1,2,\ldots,N$.

The Cartesian product of spectral sets $\{\sigma(T_k)\}_{k=1}^N$ is contained in some domain G and

$$\prod_{k=1}^N \sigma(T_k) \subset \prod_{k=1}^N (U_k \cup C_k) \subset G,$$

where $f(\zeta_1,\zeta_2,\ldots,\zeta_N)$ is a holomorphic function on $\bar{D}(f) \supset G$, $D(f)$ is a domain of $f(\zeta_1,\zeta_2,\ldots,\zeta_N)$. If X is a complex Hilbert space, this integral will be reduced to the following form*

$$f(T_1,T_2,\ldots,T_N) = \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\cdots\int_{\alpha_N}^{\beta_N} f(\lambda_1,\lambda_2,\ldots,\lambda_N)dE_1(\lambda)dE_2(\lambda)\ldots dE_N(\lambda).$$

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†Reference 1.

*This proof will be given in Theorem 2.

The operator $f(T_1,T_2,\ldots,T_N)$ is an extension of the Dunford Integral

$$f(T) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-T}\,d\zeta.$$

Another generalization of this integral is given as follows:

$$f(T_1 \otimes T_2) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-T_1 \otimes T_2}\,d\zeta,$$

where $T_1,T_2 \in B(X,X)$ and $T_1 \otimes T_2$ is a tensor product of two operators.

The main purpose of this paper is to discuss spectra of these operators.

1. Notation.

$\sigma(T)$: spectral set of an operator T.

$\rho(T)$: resolvent set.

$R_\zeta(T)$: resolvent operator $(\zeta I-T)^{-1}$.

$\prod_{k=1}^{N} \sigma(T_k) = \sigma(T_1)\times\sigma(T_2)\times\ldots\times\sigma(T_N)$: Cartesian product of spectral sets.

$B(X,X)$: set of all bounded linear operators of a complex Banach space X into the Banach space X.

$T_1 \otimes T_2$: tensor product of two operators.

$\{E_\lambda\}$: resolution of Identities.

$H[\bar{D}]$: the set of all holomorphic functions on $\bar{D}$.

$A[T_1,T_2,\ldots,T_N] = \{f(T_1,T_2,\ldots,T_N) \mid f \in H[\bar{D}],\ T_k \in B(X,X)\ (k = 1,2,\ldots N)\}$.

CHAPTER II

SOME BASIC RESULTS

1. Basic Definitions and Properties of an Operator $f(T)$.

In order to describe the Dunford integral and its generalization, it is important to know the structure of the spectral set and the resolvent operator.

Definition 1. By an analytic extension of $R_\zeta(T)\,x$ we will mean a function f defined and holomorphic on an open set $D(f) \supset \rho(T)$ and such that

$$(\zeta I-T)f(\zeta) = x, \qquad T\in B(X,X)$$

for all $\zeta\in D(f)$.

It is clear that the extension exists since

$$f(\zeta) = (\zeta I-T)^{-1}x \qquad \text{for } \zeta\in\rho(T).$$

Definition 2. The function $R_\zeta(T)x = (\zeta I-T)^{-1}x$ is said to have single-valued extension property provided that for every pair f, g of analytic extensions of $R_\zeta(T)x$, we have $f(\zeta) = g(\zeta)$ for every $\zeta\in D(f) \cap D(g)$. The union of the sets $D(f)$ as f varies over all analytic extension of $R_\zeta(T)x$ is called the resolvent set of x and is denoted by $\rho(x)$. The spectrum $\sigma(x)$ of x is defined by $\sigma(x) = \mathbb{C}-\rho(x)$, where $\mathbb{C}$ is a complex plane.

We have the following property:

<u>Proposition 1</u>. If $R_\zeta(T)x$ has the single-valued extension property, then there is a maximal extension $x(\cdot)$ whose domain is $\rho(x)$.

Proof. Let $\{f_\alpha:\alpha\in A\}$ be a class of analytic extensions of $R_\zeta(T)x$, then $\{f_\alpha:\alpha\in A\}$ is not empty since

$$(\zeta I-T)f(\zeta) = x, \qquad \zeta\in\rho(T),$$

and $f(\zeta) = R_\zeta(T)x$ is analytic on the set $\rho(T)$.

For two arbitrary functions f_α, $f_\beta\in\{f_\alpha:\alpha\in A\}$, we have

$$(\xi I-T)f_\alpha(\xi) = x \qquad \text{for any } \xi\in D(f_\alpha)\supset\rho(T)$$

$$(\eta I-T)f_\beta(\eta) = x \qquad \text{for any } \eta\in D(f_\beta)\supset\rho(T),$$

and

$$f_\alpha(\xi) = f_\beta(\xi) \qquad \text{if } \xi\in D(f_\alpha)\cap D(f_\beta)$$

since $R_\zeta(T)x$ has the single valued extension property.

Defining a function f such that

$$f = f_\alpha\cdot\chi_{D(f_\alpha)}+f_\beta\cdot\chi_{D(f_\beta)-D(f_\alpha)},$$

where χ is a characteristic function, we have

$$f = \begin{cases} f_\alpha & \text{if } \xi\in D(f_\alpha) \\ f_\beta & \text{if } \xi\in D(f_\beta)-D(f_\alpha) \end{cases} \qquad f > f_\alpha,\ f_\beta.^\dagger$$

This function is defined uniquely on $D(f_\alpha)\cup D(f_\beta)$ such that

$$(\xi I-T)f(\xi) = x \qquad \text{if } \xi\in D(f_\alpha)\cup D(f_\beta).$$

Therefore, f is an unique analytic extension of $R_\zeta(T)x$ on

$$D(f_\alpha)\cup D(f_\beta)\supset\rho(T).$$

Let f be an analytic extension of $R_\zeta(T)\,x$ whose domain is

$$\bigcup_{\alpha<\beta} D(f_\alpha),$$

that is

$$(\zeta I-T)f(\zeta) = x \qquad \text{for any } \zeta\in\bigcup_{\alpha<\beta} D(f_\alpha)\supset\rho(T)$$

and

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† $f > f_\alpha$ means that f_α precedes f.

$$f_\beta \in \{f_\alpha | \alpha \in A\}.$$

We then can define a function g such that

$$g = f \cdot \chi_{\bigcup_{\alpha<\beta} D(f_\alpha)} + f_\beta \cdot \chi_{D(f_\beta) - \bigcup_{\alpha<\beta} D(f_\alpha)}, \quad g > f, f_\beta.$$

Hence

$$(\xi I - T) g(\xi) = x \qquad \text{for any } \xi \in \bigcup_{\alpha \le \beta} D(f_\alpha) \supset \rho(T)$$

and we have a maximal analytic extension $x(\cdot)$ of $R_\zeta(T)\, x$ defining a chain as follows:

$$\begin{array}{ccccccccc} D(f_\alpha) \subset \cdots \subset & \bigcup_{\alpha<\beta} D(f_\alpha) & \subset & \bigcup_{\alpha\le\beta} D(f_\alpha) & \subset \cdots \subset & \bigcup_{\alpha\in A} D(f_\alpha) \\ \updownarrow & \updownarrow & & \updownarrow & & \updownarrow \\ f_\alpha < \cdots < & f & < & g & < \cdots < & x(\cdot). \end{array}$$

The extension always exists since $\bigcup_{\alpha\in A} D(f_\alpha)$ is a maximal set containing $\rho(T)$, and is a maximal extension of $R_\zeta(T)x$ whose domain is $\bigcup_{\alpha\in A} D(f_\alpha)$. This completes the proof.

For any $\xi_1, \xi_2 \in \rho(T)$, $\qquad f(\xi_i) = R_{\xi_i}(T)\, x.$

Since $f_\alpha(\xi) = f_\beta(\xi) \quad \xi \in \rho(T), \quad f_\alpha(\xi_1) \neq f_\beta(\xi_2) \quad$ for $\xi_1 \neq \xi_2$,

whence

$$f_\alpha(\xi_1) \neq f_\alpha(\xi_2) \qquad \text{if } \xi_1 \neq \xi_2.$$

$$\frac{f(\xi_1) - f(\xi_2)}{\xi_1 - \xi_2} = -\frac{x}{(\xi_1 I - T)(\xi_2 I - T)}$$

$$\therefore f'(\xi) = -\frac{x}{(\xi I - T)^2}$$

since $R_\zeta(T)$ is analytic (holomorphic), so is $R_\zeta(T)^2$ and $f'(\xi)$ is

analytic, whence we have

$$f^{(n)}(\xi) = (-1)^n n! R_\xi(T)^n \cdot f(\xi).$$

(i) The resolvent operator $R_\zeta(T) = (\zeta I - T)^{-1}$ is continuous for all $\zeta \in \rho(T)$, so $\rho(T) \supset \{\zeta \in \mathbb{C} : |\zeta| > \|T\|\}$ and the spectral radius $r(T)$ is represented by

$$r(T) = \sup_{\zeta \in \sigma(T)} |\zeta| = \lim_{n \to \infty} \sqrt[n]{\|T^n\|}.$$

If $\|T_\alpha\| \leq \|T\|$, then $r(T_\alpha) \leq r(T)$, whence

$$\sigma(T_\alpha) \subset \sigma(T)$$

$$\therefore \quad \sigma(T) \cap \sigma(T_\alpha) \neq \phi \qquad \text{for } T,\ T_\alpha \in B(X,X).$$

(ii) For any T, $T_1 \in B(X,X)$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|R_\zeta(T) - R_\zeta(T_1)\| < \varepsilon \qquad \text{if } \|T - T_1\| < \delta, \quad \zeta \in \rho(T) \cap \rho(T_1).$$

That is, the resolvent operator is continuous for T with respect to the uniform operator topology:

For,

$$R_\zeta(T\ ) = \frac{1}{\zeta - T_1} = \frac{1}{\zeta - T} \cfrac{1}{1 - \cfrac{T_1 - T}{\zeta - T}}$$

$$= \frac{1}{\zeta - T} \sum_{n=0}^{\infty} \left(\frac{T_1 - T}{\zeta - T}\right)^n = R_\zeta(T) \sum_{n=0}^{\infty} [(T_1 - T) R_\zeta(T)]^n$$

whence

$$\|R_\zeta(T_1) - R_\zeta(T)\| = \|R_\zeta(T)(T_1 - T) R_\zeta(T)[1 + (T_1 - T) R_\zeta(T) + \ldots]\|$$

$$= \left\|R_\zeta(T)(T_1 - T) R_\zeta(T) \frac{1}{1 - (T_1 - T) R_\zeta(T)}\right\| \leq M^2 \|T_1 - T\| (1 - \|T_1 - T\| M)^{-1}$$

where

$$M = \|R_\zeta(T)\|.$$

Putting

$$\|T_1-T\| < \frac{\varepsilon}{M^2} + \varepsilon M = \delta$$

$$\|R_\zeta(T_1)-R_\zeta(T)\| < \frac{M^2\cdot\varepsilon/M^2+\varepsilon M}{1-M\cdot\varepsilon/M^2+\varepsilon M} < \varepsilon.$$

(iii) The resolvent operator $R_\zeta(T)$ is represented by a series

$$R_\zeta(T) = \sum_{n=1}^{\infty} T^n \zeta^{-(n+1)}$$

and is uniformly convergent for all $\zeta\epsilon\{\zeta: |\zeta| > r_\sigma(T)\}$.

Let C be a s.c.o.r.c. of a boundary of an open set U such that $U \supset \sigma(T)$. Then it is obvious that

$$\frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta^n} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \neq 1, \end{cases} \qquad \frac{1}{2\pi i}\int_C \frac{T^{k-1}}{\zeta^k}\, d\zeta = 0 \quad (k > 1)$$

and

$$I = \frac{1}{2\pi i}\int_C R_\zeta(T)d\zeta.$$

Therefore 1 corresponds to an Identity operator I, and

$$\frac{1}{2\pi i}\int_C \zeta^p R_\zeta(T)d\zeta = \frac{T^p}{2\pi i}\int_C \frac{d\zeta}{\zeta} = T^p \qquad \text{for } p = 0,1,2,\ldots,$$

i.e., ζ^p corresponds to T^p. Hence we have the following:

<u>Proposition 2</u>. A polynomial $p(\zeta) = a_0+a_1\zeta+a_2\zeta^2+\ldots+a_n\zeta^n$ corresponds to a polynomial of an operator $p(T) = a_0I+a_1T+\ldots+a_nT^n$, and

$$p(T) = \frac{1}{2\pi i}\int_C \frac{p(\zeta)}{\zeta-T}\, d\zeta.$$

It is reasonable to have the following definition. For a holomorphic function $f\epsilon H[\bar{D}]$, $D \supset C \cup U$, we define

$$f(T) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-T}\,d\zeta, \qquad T\in B(X,X).$$

(iv) Let $T\in B(X,X)$ and $f\in H[D]$, $D\supset U\cup C$, $U\supset\sigma(T)$ then $f(T)$ is continuous with respect to T, that is, for any $\varepsilon>0$, there is a $\delta>0$ such that

$$\|f(T)-f(T_1)\| < \varepsilon \qquad \text{for } \|T-T_1\| < \delta.$$

Proof. According to the argument (ii), $R_\zeta(T_\alpha)$ is convergent uniformly to $R_\zeta(T)$ if $\zeta\in C$.

$$\|f(T_\alpha)-f(T)\| = \frac{1}{2\pi}\left\|\int_C f(\zeta)\{R_\zeta(T_\alpha)-R_\zeta(T)\}d\zeta\right\|$$

$$\le \frac{1}{2\pi}\int_C |f(\zeta)|\,\|R_\zeta(T_\alpha)-R_\zeta(T)\|\,|d\zeta|$$

$$\le \frac{1}{2\pi}\max_{\zeta\in C}|f(\zeta)|\cdot\varepsilon'\cdot\ell(C) = \frac{1}{2\pi}M'\cdot\varepsilon'\cdot\ell(C) = \varepsilon$$

whenever $M = \|R_\zeta(T)\|$ and $\|T_\alpha-T\| < \varepsilon'/M^2+\varepsilon'M = \delta'$.

(v) Let us consider $T_\alpha\to T$, that is, T_α converges to T with respect to the uniform operator topology and $\|T_\alpha-T\|<\delta$, then

$$f(T_\alpha)-f(T) = \frac{1}{2\pi i}\int_C f(\zeta)[R_\zeta(T_\alpha)-R_\zeta(T)]d\zeta$$

$$= \frac{1}{2\pi i}\int_C f(\zeta)\,\frac{\zeta-T-(\zeta-T_\alpha)}{(\zeta-T_\alpha)(\zeta-T)}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_C f(\zeta)\,\frac{T_\alpha-T}{(\zeta-T_\alpha)(\zeta-T)}\,d\zeta$$

$$\frac{f(T_\alpha)-f(T)}{T_\alpha-T} = \frac{1}{2\pi i}\int_C f(\zeta)\,\frac{d\zeta}{(\zeta-T_\alpha)(\zeta-T)}.$$

On the other hand,

$$\left\| \int_C \frac{f(\zeta)}{(\zeta I-T_\alpha)(\zeta I-T)}\, d\zeta - \int_C \frac{f(\zeta)}{(\zeta I-T)^2}\, d\zeta \right\|$$

$$\leq M\left\| \int_C \frac{T_\alpha - T}{(\zeta I-T_\alpha)(\zeta I-T)^2}\, d\zeta \right\|, \qquad M = \max_{\zeta\in C}|f(\zeta)|,$$

$$\leq M\|T_\alpha - T\| \frac{\ell(C)}{(\|\zeta I\|-\|T_\alpha\|)(\|\zeta I\|-\|T\|)^2} \qquad \begin{array}{l}\ell(C) = \text{length of } C\\ \|I\| = 1\end{array}$$

$$= M\ell(C)\|T_\alpha - T\| \frac{1}{(|\zeta| - T_\alpha)(|\zeta| - T)^2} \longrightarrow 0.$$

Consequently, we have

$$\lim_{T_\alpha \to T} \frac{f(T_\alpha)-f(T)}{T_\alpha - T} = f'(T) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta I-T)^2}\, d\zeta$$

and

$$\|f'(T)\| = \frac{1}{2\pi}\left\|\int_C \frac{f(\zeta)}{(\zeta-T)^2}\, d\zeta\right\| \leq \frac{1}{2\pi} M\cdot N^2\cdot \ell(C)$$

$$N = \|\zeta - T\|, \qquad M = \max_{\zeta\in C}|f(\zeta)|$$

whence $f'(T)$ certainly exists and is bounded.

(vi) We assume that the characteristic of an operator $(\zeta I-T)$ is 0. Now we shall prove that

$$f^{(n)}(T) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta I-T)^{n+1}}\, d\zeta, \qquad (n = 1,2,3,\ldots).$$

Assume that

$$f^{(k)}(T) = \frac{k!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta I-T)^{k+1}}\, d\zeta.$$

Now all we have to do is to show this is valid for k+1. Let

$\|T_\alpha - T\| < \delta$,

$$f^{(k)}(T_\alpha)-f^{(k)}(T) = \frac{k!}{2\pi i}\,[(\int_C \frac{1}{(\zeta-T_\alpha)^{k+1}} - \frac{1}{(\zeta-T)^{k+1}})f(\zeta)]d\zeta$$

$$= \frac{k!}{2\pi i}\,[\int_C \frac{(\zeta-T)^{k+1}-(\zeta-T_\alpha)^{k+1}}{(\zeta-T_\alpha)^{k+1}(\zeta-T)^{k+1}}\, f(\zeta)]d\zeta.$$

According to the Binomial Theorem, we have

$$(\zeta-T)^{k+1}-(\zeta-T_\alpha)^{k+1} = \binom{k+1}{1}\zeta^k(T_\alpha-T)-\binom{k+1}{2}\zeta^{k-1}(T_\alpha^2-T^2)+\ldots+(-1)^k(T_\alpha^{k+1}-T^{k+1})$$

$$= (k+1)(T_\alpha-T)[\zeta^k-\frac{k}{2!}\,\zeta^{k-1}(T_\alpha+T)+\ldots+(-1)^k(T_\alpha^k+T_\alpha^{k-1}T\ldots+T^k)].$$

Hence we get

$$\frac{f^{(k)}(T_\alpha)-f^{(k)}(T)}{T_\alpha-T}$$

$$= \frac{(k+1)!}{2\pi i}\int_C \frac{[\zeta^k-\frac{k}{2!}\,\zeta^{k-1}(T_\alpha+T)+\ldots+(-1)^k(T_\alpha^k+\ldots+T^k)]f(\zeta)}{(\zeta-T_\alpha)^{k+1}(\zeta-T)^{k+1}}\,d\zeta.$$

Therefore we have

$$f^{(k+1)}(T) = \frac{(k+1)!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-T)^{k+2}}\,d\zeta.$$

(vii) It is well known that the mapping

$$\psi: f(\zeta) \to f(T)$$

defined by the integral

$$f(T) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta I-T}\,d\zeta$$

satisfies the following relations:

(1) $\psi(f+g) = \psi(f)+\psi(g)$

(2) $\psi(f\cdot g) = \psi(f)\cdot\psi(g)$

(3) $\psi(\alpha f) = \alpha\psi(f)$

for any f, $g\in H[\bar{D}]$ and any complex number α. It is obvious that, if $f(\zeta)\in H[\bar{D}]$, then so is $f^{(n)}(\zeta)\in H[\bar{D}]$ for all natural numbers n. Therefore

$$\psi(f^{(n)}(\zeta)) = f^{(n)}(T),$$

that is,

$$f^{(n)}(T) = \frac{1}{2\pi i}\int_C \frac{f^{(n)}(\zeta)}{\zeta I-T}\, d\zeta.$$

Hence by (vi), we have

$$f^{(n)}(T) = \frac{1}{2\pi i}\int_C \frac{f^{(n)}(\zeta)}{\zeta I-T}\, d\zeta = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta I-T)^{n+1}}\, d\zeta$$

$$\text{i.e.,} \quad \int_C \{\frac{d^n}{d\zeta^n} f(\zeta)\}R_\zeta(T)d\zeta = \int_C f(\zeta)\{\frac{\partial^n}{\partial T^n} R_\zeta(T)\}d\zeta.$$

(viii) The formula

$$f(T+S) = \sum_{n=0}^{\infty} \frac{f^{(k)}(S)}{k!}\, T^k$$

is well known,[†] but it is proved from a quite different point of view:

Let T, $S\in B(X,X)$ with $\|T\| < |\zeta|+\|S\|$ for any ζ such that $\zeta > \|T+S\|$, and let f be a holomorphic function on $D \supset C \cup U$, $U \supset \sigma(T+S)$, then we have

$$f(T+S) = \sum_{n=0}^{\infty} \frac{f^{(k)}(S)}{k!}\, T^k.$$

Proof. Since S, $T\in B(X,X)$, $T+S\in B(X,X)$, the integral,

$$f(T+S) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta I-(T+S)}\, d\zeta.$$

The resolvent operator can be represented by a uniformly convergent series such that

$$R_\zeta(T+S) = \frac{1}{(\zeta I-S)-T} = \sum_{k=0}^{\infty} \frac{T^k}{(\zeta I-S)^{k+1}} \qquad \text{for } \zeta\in C.$$

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† See Reference 4 and 5.

Whence

$$\int_C \frac{f(\zeta)}{\zeta-(T+S)}\,d\zeta = \int_C f(\zeta)\cdot\sum_{k=0}^{\infty} \frac{T^k}{(\zeta I-S)^{k+1}}\,d\zeta$$

$$= \sum_{k=0}^{\infty} T^k \int_C \frac{f(\zeta)}{(\zeta I-S)^{k+1}}\,d\zeta \quad .$$

According to (vii),

$$f(T+S) = \sum_{k=0}^{\infty} \frac{T^k}{k!}\left\{\frac{k!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta I-S)^{k+1}}\,d\zeta\right\}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(S)}{k!}\,T^k.$$

Putting $S = \Theta$ (zero operator), we get

$$f(T) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!}\,T^k.$$

This can be regarded as a Taylor expansion of the operator around zero. Replacing $T = I$ (Identity operator), we have

$$f(I) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!}\,I^k = \left\{\sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!}\right\}I = f(1)\cdot I,$$

where

$$f(\Theta) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta I-\Theta}\,d\zeta = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-0}\,d\zeta = f(0),$$

so we can identify $f(\Theta)$ with $f(0)$.

(ix) We have defined $f(T) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-T}\,d\zeta$. Since $\frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta-T} = I$, I is an Identity operator, we have

$$f(T) = f(T)I = f(T)\cdot\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-T}.$$

This relation suggests that

$$f(T)\cdot \frac{1}{2\pi i}\int_C \frac{d\zeta}{\zeta-T} = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-T}\,d\zeta.$$

More generally we get the following property:

Let $\{\sigma_k\}_{k=1}^n$ be subsets of $\sigma(T)$ such that

(1) $\sigma_k \cap \sigma_j = 0$ if $k \neq j$

(2) $\bigcup_{k=1}^n \sigma_k = \sigma(T)$

(3) There exist s.c.o.r.c.'s $\{C_k\}_{k=1}^n$ such that $C_k \cap C_j = 0$ if $k \neq j$ and $C_k \cup U_k \supset \sigma_k$, C_k lies in $\rho(T)$ for each k. Then

$$E(\sigma_k)f(T) = f(T)E(\sigma_k) = \frac{1}{2\pi i}\int_{C_k} \frac{f(\eta)}{\eta-T}\,d\eta$$

where

$$E(\sigma_k) = \frac{1}{2\pi i}\int_{C_k} \frac{d\eta}{\eta-T}.$$

Proof. We shall prove this by using the expansion which is described in (viii).

$$f(T)E(\sigma_k) = f(T)\cdot \frac{1}{2\pi i}\int_{C_k} \frac{d\eta}{\eta-T}$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!}\, T^k\, \frac{1}{2\pi i}\int_{C_k} \frac{d\eta}{\eta-T}, \qquad T^k = \frac{1}{2\pi i}\int_C \frac{\zeta^k}{\zeta-T}\,d\zeta$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!}\, \frac{1}{(2\pi i)^2}\int_C \frac{\zeta^k}{\zeta-T}\,d\zeta\cdot\int_{C_k} \frac{d\eta}{\eta-T}$$

$$= \frac{1}{(2\pi i)^2}\sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!}\int_C\int_{C_k} \frac{\zeta^k}{(\zeta-T)(\eta-T)}\,d\zeta d\eta.$$

Since $R_\zeta(T)\cdot R_\eta(T) = \frac{1}{\eta-\zeta}(R_\zeta(T)-R_\eta(T))$,

$$f(T)E(\sigma_k) = \frac{1}{(2\pi i)^2}\sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!}\left(\int_C\int_{C_k} \frac{\zeta^k}{\eta-\zeta}\,(R_\zeta(T)-R_\eta(T))\,d\zeta d\eta\right)$$

$$= \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!} \left(\int_C \int_{C_k} \frac{\zeta^k}{\eta-\zeta} R_\zeta(T) d\zeta d\eta - \int_C \int_{C_k} \frac{\zeta^k}{\eta-\zeta} R_\eta d\zeta d\eta \right)$$

$$= \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!} \left(\int_C R_\zeta \zeta^k d\zeta \int_{C_k} \frac{d\eta}{\eta-\zeta} - \int_{C_k} R_\eta d\eta \int_C \frac{\zeta^k}{\eta-\zeta} d\zeta \right)$$

where

$$\int_{C_k} \frac{d\eta}{\eta-\zeta} = 0, \qquad \int_C \frac{\zeta^k}{\zeta-\eta} d\zeta = 2\pi i \eta^k$$

since ζ is outside of C_k and η is inside of C. Therefore we have

$$f(T)E(\sigma_k) = \frac{1}{(2\pi i)^2} \sum_{k=0}^{\infty} \frac{f^{(k)}(\Theta)}{k!} \int_{C_k} R_\eta(T) d\eta \cdot 2\pi i \eta^k$$

$$= \frac{1}{2\pi i} \int_{C_k} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \eta^k \cdot R_\eta d\eta = \frac{1}{2\pi i} \int_{C_k} \frac{f(\eta)}{\eta - T} d\eta$$

$$\text{i.e.,} \quad f(T)E(\sigma_k) = E(\sigma_k)f(T) = \frac{1}{2\pi i} \int_{C_k} \frac{f(\eta)}{\eta-T} d\eta.$$

It is easily seen that

$$f(T)E(\sigma_i \cup \sigma_j) = \frac{1}{2\pi i} \int_{C'} \frac{f(\eta)}{\eta - T} d\eta, \qquad C' \cup U' \supset \sigma_i \cup \sigma_j$$

C' lies inside of $\rho(T)$. Hence we have

$$f(T)E\left(\bigcup_{k=1}^{n} \sigma_k \right) = f(T) \cdot E(\sigma(T)) = f(T)I = \frac{1}{2\pi i} \int_C \frac{f(\eta)}{\eta - T} d\eta$$

$$U \supset \sigma(T).$$

2. A Reduced Integral Representation of the Operator f(T).

(1) We would like to reduce the integral

$$f(T) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - T} d\zeta$$

to another form. To do this, we shall restrict our attention to a

complex Hilbert space H and consider $T\in B(H,H)$, bounded linear operator of H into H.

The expression (Tf,g) defines a bilinear functional on H, for each bilinear functional $\Omega(f,g)$ has a representation of the form

$$\Omega(f,g) = (Tf,g)$$

where $T\in B(H,H)$ is uniquely determined by the bilinear functional Ω, furthermore

$$\|\Omega\| = \|T\|.^*$$

Let $T\in B(H,H)$, and assume that $T = T^*$, then the following statements are obvious:

(i) (Tx,x) is real.

(ii) Eigen value of T is real.

(iii) T is continuous.

(iv) Putting $m(T) = \inf\limits_{\|x\|=1}(Tx,x)$, $M(T) = \sup\limits_{\|x\|=1}(Tx,x)$.

(v) $m(T) \leq M(T)$ and $m(T)$, $M(T) < \infty$ since T is bounded.

(vi) $m(T) \leq \lambda \leq M(T)$, λ is any eigen value of T.

(vii) There exist orthogonal projections $\{E_\lambda\}$ defined for each λ with

(a) $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$ if $\lambda \leq \mu$

(b) $\lim\limits_{\mu \to \lambda^+} E_\mu x = E_\lambda x$

(c) $E_\lambda = 0$ if $\lambda < m(T)$, $E_\lambda = I$ if $\lambda \geq M(T)$

(d) $E_\lambda T = TE_\lambda$.

For this family of one parameter of projections $\{E_\lambda\}$, the formula

$$p(T) = \int_\alpha^\beta p(\lambda)\,dE_\lambda$$

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* See Reference 7, p.42.

holds, and the integral is defined by the norm topology of the operator space B(H,H).

Proof.* $\alpha < m(T)$, $M(T) \le \beta$.

Suppose that

$$\alpha = \lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_n = \beta$$

and

$$\lambda_{k-1} \le \mu_k \le \lambda_k \qquad k = 1,2,\dots,n.$$

Let

$$B = \sum_{k=1}^{n} p(\mu_k)[E(\lambda_k)-E(\lambda_{k-1})] = \sum_{k=1}^{n} \int_{\lambda_{k-1}}^{\lambda_k} p(\mu_k)dE_\lambda$$

$$(Bx,x) = \sum_{k=1}^{n} \int_{\lambda_{k-1}}^{\lambda_k} p(\mu_k)d(E_\lambda x,x).$$

Putting

$$(p(T)x,x) = \sum_{k=1}^{n} \int_{\lambda_{k-1}}^{\lambda_k} p(\lambda)d(E_\lambda x,x),$$

we have

$$(p(T)x,x)-(Bx,x) \le \sum_{k=1}^{n} \int_{\lambda_{k-1}}^{\lambda_k} \varepsilon\cdot d(E_\lambda x,x) = \varepsilon(x,x)$$

where

$$\varepsilon_k = \max|p(\lambda)-p(\mu)|, \qquad \lambda,\mu\in[\lambda_{k-1},\lambda_k], \qquad \text{and}$$

$$\varepsilon = \max\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n\}.$$

Hence

$$((p(T)-B)x,x) \le \varepsilon\cdot(x,x)$$

and

$$\sup_{\|x\|=1} ((p(T)-B)x,x) \le \varepsilon \qquad \text{i.e.,} \quad M\|p(T)-B\| \le \varepsilon.$$

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* See Reference 6, p.350.

Similarly,

$$-\varepsilon \le m[p(T)-B] \le \varepsilon$$

$$\therefore \|p(T)-B\| < \varepsilon.$$

Therefore we get

$$p(T) = \int_{\alpha}^{\beta} p(\lambda)\,dE_{\lambda}. \tag{2-1}$$

(2) We shall show that the formula (2-1) can be reduced by the Dunford integral. More generally we have the following:

Proposition 3. Let $T \in B(H,H)$, $T = T^*$, then we have

$$f(T) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - T}\,d\zeta = \int_{\alpha}^{\beta} f(\lambda)\,dE_{\lambda}; \tag{2-2}$$

α, β are some finite real numbers.

Proof.

$$R_{\zeta}(T) = (\zeta - T)^{-1} = \sum_{k=0}^{\infty} \zeta^{-(k+1)} T^k$$

is uniformly convergent since $\zeta \in C$, $|\zeta| > \|T\|$. Therefore

$$f(T) = \frac{1}{2\pi i}\int_C \sum_{k=0}^{\infty} f(\zeta)\,\frac{d\zeta}{(\zeta-0)^{k+1}}\,T^k$$

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty} T^k \int_C \frac{f(\zeta)}{(\zeta-0)^{k+1}}\,d\zeta$$

$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\,T^k = \sum_{k=0}^{\infty}\int_{\alpha}^{\beta} \frac{f^{(k)}(0)}{k!}\,\lambda^k dE_{\lambda} \quad \text{(by 2-1)}$$

$$= \int_{\alpha}^{\beta} f(\lambda)\,dE_{\lambda}$$

where $\alpha = \min\{\lambda\ C \cap R\}$ and $\beta = \max\{\lambda\ C \cap R\}$. R is a real axis.

CHAPTER III

THE SPECTRUM OF AN OPERATOR $f(T_1,T_2,\ldots,T_N)$

1. The Generalized Spectral Mapping Theorem and a Reduced Formula of $f(T_1,T_2,\ldots,T_N)$.

Let $T_k \in B(X,X)$ $k = 1,2,\ldots,N$, and let f be a holomorphic function on $\bar{D}(f)$, that is, $f \in H[\bar{D}]$, where

$$\bar{D}(f) \supset (U_1 \cup C_k) \times (U_2 \cup C_k) \times \ldots \times (U_N \cup C_N).$$

$U_k \supset \sigma(T_k)$ $(k = 1,2,\ldots N)$, and each C_k is a s.c.o.r.c. lying in the resolvent set $\rho(T_k)$ of T_k, then we define an operator

$$f(T_1,T_2,\ldots,T_N) = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\ldots\int_{C_N} f(\xi_1,\xi_2,\ldots,\xi_N) \tag{1-1}$$

$$\times R_{\xi_1}(T_1)R_{\xi_2}(T_2)\ldots R_{\xi_N}(T_N)\cdot d\xi_1 d\xi_2\ldots d\xi_N.$$

First we shall prove the following:

Lemma 1. An operator $f(T_1,T_2,\ldots,T_N)$ is independent on the simple closed oriented rectifiable curves C_k $(k = 1,2,\ldots N)$, dependent only on the function $f \in H[\bar{D}]$ and operators $T_k \in B(X,X)$ $k = 1,2,\ldots,N$.

Proof. Since $g(\zeta_k) = (\xi_k-\zeta_k)^{-1}$, $(\xi_k \neq \zeta_k)$ is a holomorphic function, $R_{\xi_k}(T_k)$ can be represented by

$$R_{\xi_k}(T_k) = \frac{1}{2\pi i}\int_{K_k} \frac{R_{\zeta_k}(T_k)}{\xi_k-\zeta_k}\, d\zeta_k \quad \dagger \tag{1-2}$$

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† See Reference 2, p.196.

where K_k is a s.c.o.r.c. and

$$K_k \supset \sigma(T_k)$$

for each k.

In the first case, we suppose that K_k lies inside of C_k that is, for two T-admissible domain[†] D_k and D_k^* such that

$$B(D_k) = C_k, \quad B(D_k^*) = K_k \quad \text{and} \quad D_k \supsetneq D_k^* \supsetneq \sigma(T_k),$$

let

$$C_k \cup D_k \supset K_k \cup D_k^*.$$

We have

$$f(T_1,T_2,\ldots,T_N) = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{K_1}\cdots\int_{C_N}\int_{K_N} f(\xi_1,\xi_2,\ldots,\xi_N)$$

$$\times\ (\xi_1-\zeta_1)^{-1}(\xi_2-\zeta_2)^{-1}\ldots(\xi_N-\zeta_N)^{-1}R_{\zeta_1}(T_1)R_{\zeta_2}(T_2)\ldots R_{\zeta_N}(T_N)d\xi_1 d\zeta_1\ldots d\xi_N d\zeta_N$$

since

$$\int_{C_k} f(\xi_1,\xi_2,\ldots,\xi_N)/\xi_k-\zeta_k \cdot d\xi_k = 2\pi i f(\xi_1,\xi_2,\ldots,\xi_{k-1},\zeta_k,\xi_{k+1},\ldots\xi_N)$$

$$k = 1,2,\ldots,N, \qquad \xi_k \neq \zeta_k;$$

whence we have

$$\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\xi_1,\xi_2,\ldots,\xi_N)R_{\xi_1}(T_1)R_{\xi_2}(T_2)\ldots R_{\xi_N}(T_N)d\xi_1 d\xi_2\ldots d\xi_N$$

$$= \int_{K_1}\int_{K_2}\cdots\int_{K_N} f(\zeta_1,\zeta_2,\ldots,\zeta_N)R_{\zeta_1}(T_1)R_{\zeta_2}(T_2)\ldots R_{\zeta_N}(T_N)d\zeta_1 d\zeta_2\ldots d\zeta_N.$$

In the second case, let $D_k \not\supset D_k^*$ or $D_k^* \not\supset D_k$. Since $D_k \cap D_k^*$ is open,

$$D_k \cap D_k^* \supsetneq \sigma(T_k).$$

[†]See Reference 2, p.192.

It is a well-known fact that $\sigma(T_k)$ is not empty and is closed for each $k = 1,2,\ldots N$. The complex plane is a metric space with the usual metric, so is a T_4-space. Therefore there exists an open set D_1 such that

$$D \cap D^* \supsetneqq \bar{D}_1 \supset \sigma(T_k).$$

Putting $B(D_1) = S_k$, we have

$$\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\xi_1,\xi_2,\ldots,\xi_N)R_{\xi_1}(T_1)R_{\xi_2}(T_2)\ldots R_{\xi_N}(T_N)d\xi_1 d\xi_2\ldots d\xi_N$$

$$= \int_{S_1}\int_{S_2}\cdots\int_{S_N} f(\eta_1,\eta_2,\ldots,\eta_N)R_{\eta_1}(T_1)R_{\eta_2}(T_2)\ldots R_{\eta_N}(T_N)d\eta_1 d\eta_2\ldots d\eta_N$$

$$= \int_{K_1}\int_{K_2}\cdots\int_{K_N} f(\zeta_1,\zeta_2,\ldots,\zeta_N)R_{\zeta_1}(T_1)R_{\zeta_2}(T_2)\ldots R_{\zeta}(T_N)d\zeta_1 d\zeta_2\ldots d\zeta_N,$$

according to the first case. This completes the proof.

Remark 1. If $T \in B(X,X)$ and X is a complex Banach space, $\sigma(T)$ is non-empty.

For, $\|R_\zeta\| \leq (|\zeta| - \|T\|)^{-1}$ if $|\zeta| > T$, whence $\|R_\zeta\| \to 0$ as $|\zeta| \to \infty$. If $\sigma(T)$ were empty, then it would follow that $R_\zeta(T)$ is holomorphic and bounded on the whole complex plane . But it would be constant by Liouville's theorem. This is impossible due to the fact that $R_\zeta(T)$ is a one-to-one mapping of X onto X, and that X is not empty.

Remark 2. The spectrum $\sigma(T)$ is closed, that is, the resolvent set $\rho(T)$ is open.

For, if $\mu \in \rho(T)$, then $(\mu I - T)^{-1}$ exists, continuous and the closure of the range $\mu I - T$ is X. If λ is sufficiently near μ, $(\lambda I - T)^{-1}$ exists, continuous and the range of $\lambda I - T$ is X. Hence $\lambda \in \rho(T)$, this means that $\rho(T)$ is open, so $\sigma(T)$ is closed.

Definition 1-1. A mapping ϕ from a ring R into a ring R' is said to be a homomorphism if

(1) $\phi(a+b) = \phi(a)+\phi(b)$

(2) $\phi(a\cdot b) = \phi(a)\cdot\phi(b)$ for all $a,b\in R$.

Definition 1-2. A ring R is said to be an algebra over a field F if R is a vector space over F such that for all $a,b\in R$ and $\alpha\in F$, the equality

$$\alpha(ab) = (\alpha a)b = a(\alpha b)$$

holds.

Definition 1-3. By an algebraic homomorphism between two algebras we mean a ring homomorphism which is also a linear transformation over F.

Lemma 2. We set

$$H[\bar{D}] = \{f(\xi_1,\xi_2,\ldots,\xi_N):(\xi_1,\xi_2,\ldots,\xi_N)\in\bar{D}(f)\}$$

and

$$A[T_1,T_2,\ldots,T_N] = \{f(T_1,T_2,\ldots,T_N):T_k\in B(X,X), \quad k = 1,2,\ldots,N\}.$$

Then $H[\bar{D}]$ and $A[T_1,T_2,\ldots,T_N]$ are algebras over the complex field F, moreover a mapping

$$\psi:f(\xi_1,\xi_2,\ldots,\xi_N) \to f(T_1,T_2,\ldots,T_N)$$

defined by

$$f(T_1,T_2,\ldots,T_N)= \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\xi_1,\xi_2,\ldots,\xi_N)$$

$$\times R_{\xi_1}(T_1)\cdots R_{\xi_N}(T_N)d\xi_1 d\xi_2\cdots d\xi_N$$

is an algebraic homomorphism of $H[\bar{D}]$ onto $A[T_1,T_2,\ldots,T_N]$.

It is clear that $H[\bar{D}]$ is an algebra, and ψ is linear; that is,

$$\psi(\alpha f+\beta g) = \alpha\psi(f)+\beta\psi(g).$$

It is also obvious that $A[T_1,T_2,\ldots,T_N]$ is a ring and vector space over F such that

$$\alpha f(\xi_1,\xi_2,\ldots,\xi_N)\cdot g(\eta_1,\eta_2,\ldots,\eta_N) = f(\xi_1,\xi_2,\ldots,\xi_N)\cdot\alpha\cdot g(\eta_1,\eta_2,\ldots,\eta_N).$$

Therefore all we have to do is to prove that

$$\psi\{f_1(\xi_1,\xi_2,\ldots,\xi_N)\ f_2(\xi_1',\xi_2',\ldots,\xi_N')\}$$

$$= \psi\{f_1(\xi_1,\xi_2,\ldots,\xi_N)\}\cdot\psi\{f_2(\xi_1',\xi_2',\ldots,\xi_N');$$

$$f_1(T_1,T_2,\ldots,T_N)\cdot f_2(T_1,T_2,\ldots,T_N)$$

$$= \frac{1}{(2\pi i)^{2N}}\int_{C_1}\int_{C_1'}\cdots\int_{C_N}\int_{C_N'} f_1(\xi_1,\xi_2,\ldots,\xi_N)f_2(\xi_1',\xi_2',\ldots,\xi_N')$$

$$\times R_{\xi_1}(T_1)R_{\xi_1'}(T_1)\ldots R_{\xi_N}(T_N)R_{\xi_N'}(T_N)d\xi_1 d\xi_1'\ldots d\xi_N d\xi_N'.$$

We have

$$J = \int_{C_1}\int_{C_1'} f_1(\xi_1,\xi_2,\ldots\xi_N)f_2(\xi_1',\xi_2',\ldots\xi_N')R_{\xi_1}(T_1)R_{\xi_1'}(T_2)d\xi_1 d\xi_1'$$

$$= 2\pi i\int_{C_1} f_1(\xi_1,\xi_2,\ldots,\xi_N)f_2(\xi_1,\xi_2',\ldots,\xi_N')R_{\xi_1}(T_1)d\xi_1;$$

for

$$R_{\xi_1}(T_1)R_{\xi_1'}(T_1) = \frac{1}{\xi_1'-\xi_1}\left(\frac{1}{\xi_1-T_1} - \frac{1}{\xi_1'-T_1}\right)$$

whence

$$J = \int_{C_1} f_1(\xi_1,\xi_2,\ldots,\xi_N) R_{\xi_1}(T_1) \left(\int_{C_1'} \frac{f_2(\xi_1',\xi_2',\ldots,\xi_N')}{\xi_1'-\xi_1} d\xi_1' \right) d\xi_1$$

$$- \int_{C_1}\int_{C_1'} \frac{f_1(\xi_1,\xi_2,\ldots\xi_N) f_2(\xi_1',\xi_2',\ldots\xi_N')}{\xi_1'-\xi_1} R\xi_1'(T_1) d\xi_1 d\xi_1'$$

taking C_k inside of C_k' for each k, we have

$$\int_{C_1}\int_{C_1'} \frac{f_1(\xi_1,\xi_2,\ldots,\xi_N) f_2(\xi_1',\xi_2',\ldots,\xi_N')}{\xi_1'-\xi_1} R_{\xi_1'}(T_1) d\xi_1 d\xi_1' = 0$$

and

$$\frac{1}{2\pi i}\int_{C_1'} \frac{f_2(\xi_1',\xi_2',\ldots,\xi_N')}{\xi_1'-\xi_1} d\xi_1' = f_2(\xi_1,\xi_2',\ldots,\xi_N').$$

Hence we have

$$f_1(T_1,T_2,\ldots,T_N)\cdot f_2(T_1,T_2,\ldots,T_N)$$

$$= \frac{1}{(2\pi i)^{2N-1}} \int_{C_1}\int_{C_2}\int_{C_2'}\cdots\int_{C_N}\int_{C_N'} f_1(\xi_1,\xi_2,\ldots,\xi_N) f_2(\xi_1,\xi_2',\ldots,\xi_N')$$

$$R_{\xi_1}(T_1) R_{\xi_2}(T_2) R_{\xi_2'}(T_2)\ldots R_{\xi_N}(T_N) R_{\xi_N'}(T_N) d\xi_1\cdot d\xi_2\cdot d\xi_2'\ldots d\xi_N\cdot d\xi_N'.$$

Repeating this process, we owuld have

$$f_1(T_1,T_2,\ldots,T_N)\cdot f_2(T_1,T_2,\ldots,T_N)$$

$$= \frac{1}{(2\pi i)^N} \int_{C_1}\int_{C_2}\cdots\int_{C_N} f_1(\xi_1,\xi_2,\ldots,\xi_N)\cdot f_2(\xi_1,\xi_2,\ldots,\xi_N)$$

$$\times R_{\xi_1}(T_1) R_{\xi_2}(T_2)\ldots R_{\xi_N}(T_N) d\xi_1\ldots d\xi_N.$$

This means that

$$\psi\{f_1(\xi_1,\xi_2,\ldots,\xi_N)\cdot f_2(\xi_1',\xi_2',\ldots,\xi_N')\} = f_1(T_1,T_2,\ldots,T_N)\cdot f_2(T_1,T_2,\ldots,T_N)$$

$$= \psi\{f_1(\xi_1,\xi_2,\ldots,\xi_N)\}\cdot\psi\{f_2(\xi_1',\xi_2',\ldots,\xi_N')\},$$

and

$$f_1(T_1,T_2,\ldots,T_N)\cdot f_2(T_1,T_2,\ldots,T_N)\in A[T_1,T_2,\ldots,T_N]$$

i.e., $A[T_1,T_2,\ldots,T_N]$ is an algebra. Consequently, the mapping is an algebraic homomorphism.

Corollary 1. If $f(\xi_1,\xi_2,\ldots,\xi_N) = \xi_k \quad 1 \le k \le N$, then $f(T_1,T_2,\ldots,T_N) = T_k$ and $f(\xi_1,\xi_2,\ldots,\xi_N) = \xi_i\xi_k$ corresponds to $f(T_1,T_2,\ldots,T_N) = T_iT_k$, moreover $f(\xi_1,\xi_2,\ldots,\xi_N) = 1$ corresponds to an Identity operator I.

Proof.

$$\frac{1}{(2\pi i)^{N-1}}\int_{C_1}\int_{C_2}\cdots\int_{C_{k-1}}\int_{C_{k+1}}\cdots\int_{C_N}\{\frac{1}{2\pi i}\int_{C_k}\xi_k R_{\xi_k}(T_k)d\xi_k\}$$

$$\times R_{\xi_1}(T_1)\cdots R_{\xi_{k-1}}(T_{k-1})R_{\xi_{k+1}}(T_{k+1})\cdots R_{\xi_N}(T_N)d\xi_1\cdots d\xi_{k-1}d\xi_{k+1}\cdots d\xi_N$$

$$= T_k\cdot I^{N-1} = T_k,$$

since

$$\int_{C_i} R_{\xi_i}(T_i)d\xi_i = I$$

for each i. This means that

$$\psi(\xi_k) = T_k.$$

Similarly

$$\psi(\xi_i\cdot\xi_k) = \psi(\xi_i)\psi(\xi_k) = T_iT_k.$$

In the case

$$f(\xi_1,\xi_2,\ldots,\xi_N) = 1,$$

$$\frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} R_{\xi_1}(T_1)R_{\xi_2}(T_2)\ldots R_{\xi_N}(T_N)\,d\xi_1\ldots d\xi_N = I,$$

since

$$\frac{1}{2\pi i}\int_{C_k} R_{\xi_k}(T_k)\,d\xi_k = I$$

if

$$C_k \cup U_k \supset \sigma(T_k) \qquad (k = 1,2,\ldots,N).$$

Definition 1-4. We define the spectrum $\sigma[f(T_1,T_2,\ldots,T_N)]$ of an operator $f(T_1,T_2,\ldots,T_N)$ to consist of those complex numbers λ such that $\lambda I-f(T_1,T_2,\ldots,T_N)$ has no inverse, that is

$$\sigma[f(T_1,T_2,\ldots,T_N)] = \{\lambda:\bar{\exists}(\lambda I-f(T_1,T_2,\ldots,T_N))^{-1}\}.^{\dagger}$$

The resolvent set is defined by

$$\rho[f(T_1,T_2,\ldots,T_N)] = \mathbb{C}-\sigma[f(T_1,T_2,\ldots,T_N)].$$

Now, we shall discuss the spectra of the operator $f(T_1,T_2,\ldots,T_N)$. We know that the relation

$$\sigma[f(T_1,T_2,\ldots,T_N)] \subseteq \{f(\lambda_1,\lambda_2,\ldots,\lambda_N):\lambda_k\in\sigma(T_k),\ k = 1,2,\ldots,N\}$$

holds, and we would like to get opposite inclusion. The following idea is essentially the same idea as in the proof of N. Dunford's spectral mapping theorem.

For any $f(z_1,z_2,\ldots,z_N)\in\{f(z_1,z_2,\ldots,z_N):z_k\in\sigma(T_k)\quad k = 1,2,\ldots,N\}$ and $f(\xi_1,\xi_2,\ldots,\xi_N)\in H[\bar{D}]$, we define a function

$$g(\xi_1,\xi_2,\ldots,\xi_N)$$

such that

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† See Reference 11.

$$g(\xi_1,\xi_2,\ldots,\xi_N) = [f(z_1,z_2,\ldots,z_N)-f(\xi_1,\xi_2,\ldots,\xi_N)](z_k-\xi_k)^{-1}$$

is a holomorphic function on $\bar{D}(f)$.

According to the previous Lemma and corollary the function

$$g(\xi_1,\xi_2,\ldots,\xi_N)(z_k-\xi_k) = f(z_1,z_2,\ldots,z_N)-f(\xi_1,\xi_2,\ldots,\xi_N)$$

corresponds to an operator

$$g(T_1,T_2,\ldots,T_N)(z_kI-T_k) = f(z_1,z_2,\ldots,z_N)I-f(T_1,T_2,\ldots,T_N).$$

Since $z_k \in \sigma(T_k)$, we have

$$f(z_1,z_2,\ldots,z_N) \in \sigma[f(T_1,T_2,\ldots,T_N)]$$

whence we have

$$\{f(z_1,z_2,\ldots,z_N):z_k \in \sigma(T_k) \quad (k = 1,2,\ldots,N)\} \subseteq \sigma[f(T_1,T_2,\ldots,T_N)];$$

and since

$$\{f(z_1,z_2,\ldots,z_N):z_k \in \sigma(T_k) \quad (k = 1,2,\ldots,N)\}$$

$$= f(\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_N)),$$

we have the following.

Theorem 1. For a spectral set of an operator $f(T_1,T_2,\ldots,T_N)$ we have the following equality

$$\sigma[f(T_1,T_2,\ldots,T_N)] = f(\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_N)).$$

This is an extension of the "spectral mapping theorem"

$$f(\sigma(T)) = \sigma[f(T)]$$

which is due to N. Dunford. For clarity we shall say "Generalized Spectral Mapping Theorem" instead of Theorem 1.

Using this, we will get another theorem. Before doing that let us describe some immediate consequences of this theorem.

Corollary 1. For $T_k \in B(X,X)$ $(k = 1,2,\ldots,N)$ and $f \in H[\bar{D}]$, $\sigma[f(T_1,T_2,\ldots,T_N)]$ is closed.

Proof. It is a well-known fact that the spectral set $\sigma(T_k)$ is closed and bounded in the complex plane, so it is compact with respect to the topology induced by the usual metric in the complex plane, that is, $\mathbb{C}$ is a metric space with its usual metric. Moreover, according to the Tychonoff theorem, the Cartesian product

$$\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_N)$$

is compact.

Since the function $f \in H[\bar{D}]$, such that

$$f: \mathbb{C} \times \mathbb{C} \times \ldots \times \mathbb{C} \to \mathbb{C}$$

is holomorphic on $\bar{D}(f)$, then it is continuous on $\bar{D}(f)$ (according to the definition of holomorphic function), whence

$$f(\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_N))$$

is compact. Thus

$$f(\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_N)) = \sigma[f(T_1,T_2,\ldots,T_N)]$$

is closed since $\mathbb{C}$ is Hausdorff.

Remark. In the proof of the above corollary we used these facts:

(a) A continuous image of a compact set is compact.

(b) A compact set in a Hausdorff space is closed.

(c) A metric space is Hausdorff.

(d) The product space is compact iff each factor space is compact (Tychonoff).[†]

Corollary 2. The spectral set $\sigma[f(T_1,T_2,\ldots,T_N)]$ is a non-empty set if X is a complex Banach space and $T_k \in B(X,X)$ $(k = 1,2,\ldots,N)$.

Proof. The spectral set $\sigma(T_k)$ is not empty provided that $T_k \in B(X,X)$ and that X is a complex Banach space (Chapter II, Remark). Therefore

$$\prod_{k=1}^{N} \sigma(T_k)$$

is not empty, whence

$$f\left(\prod_{k=1}^{N} \sigma(T_k)\right)$$

is not empty; that is,

$$\sigma[f(T_1,T_2,\ldots,T_N)]$$

is not empty.

It is well known that,

(*1) If the bounded self-adjoint operator T satisfies $T \geq 0$, the spectrum $\sigma(T)$ lies on the positive real line, that is

$$\sigma(T) \subset [0,\infty).$$

This is an immediate corollary of the following results.

(*2) If the bounded self-adjoint operator T satisfies $T \geq I$, then T^{-1} exists and $\|T^{-1}\| \leq 1$.

Proof. Since $T \geq I$, we have

$$\|Tf\|\|f\| \geq (If,f) \geq (f,f) = \|f\|^2$$

for all $f \in H$. And hence, in particular,

[†] See Reference 9.

$$\|Tf\| \geq \|f\|.$$

Thus $Tf = 0$ implies $f = 0$. Hence T is a one-to-one transformation from H to its range, and the transformation T^{-1} is defined on the range of T satisfies $\|T^{-1}\| \leq 1$.

Now, the range of T is dense in H. For, if there is an element f which is orthogonal to Tg, we have

$$0 = (Tf,g) = (f,Tg)$$

and hence $Tf = 0$. Thus $f = 0$ and the range of T is dense in H.

We show that the range of T is all of H. Suppose that $g \in H$ and let $\{f_n\}$ be so chosen that $Tf_n \to g$. Since

$$\|Tf_n - Tf_m\| = \|T(f_n - f_m)\| \geq \|f_n - f_m\|,$$

$\{f_n\}$ is a Cauchy sequence. Suppose that $f_n \to f$. Since T is continuous, $Tf_n \to Tf$.

On the other hand, by hypothesis $Tf_n \to g$, whence $Tf = g$. Thus the range of T is all of H.

Proof of (*1). According to (*2), if λ is strictly positive, $\lambda > 0$, then $B \geq \lambda I$ implies B^{-1} exists. Write

$$B = \lambda I + T,$$

since $T \geq 0$, $\lambda I + T \geq \lambda I$ and $(\lambda I + T)^{-1}$ exists, $-\lambda \in \sigma(T)$ for $\lambda > 0$. Hence the set $(-\infty, 0)$ belongs to the resolvent set, that is,

$$\sigma(T) \subset [0,\infty).$$

The sharper interval, in which the spectral set $\sigma(T)$ lies, is obtained from the following fact.

(*3) Suppose that $T \in B(H,H)$, $T = T^*$, then $\sigma(T)$ lies in the closed interval $[m(T), M(T)]$ in the real axis. The end points of this interval belong to $\sigma(T)$, where

$$m(T) = \inf_{\|x\|=1}(Tx,x), \qquad M(T) = \sup_{\|x\|=1}(Tx,x).$$

Now, we return to another corollary to Theorem I.

Corollary 3. Let $A_R[T_1,T_2,\ldots,T_N] \subset A[T_1,T_2,\ldots,T_N]$ be a subset of all $f(T_1,T_2,\ldots,T_N)$ which corresponds to $f_R \in H[\bar{D}]$ with real coefficients, and $T_k \in B(H,H)$, $T_k \geq 0$ $(k = 1,2,\ldots,N)$; then the spectrum $\sigma[f(T_1,T_2,\ldots,T_N)]$ is real. Moreover for $f_{R\geq 0} \in H[\bar{D}]$ with non-negative coefficients, the spectrum $\sigma[f(T_1,T_2,\ldots,T_N)]$ is non-negative. The proof is immediate.

Corollary 3 suggests that the operator $f(T_1,T_2,\ldots,T_N)$ will be represented by another form.

Theorem 2. Let $f(\xi_1,\xi_2,\ldots,\xi_N) \in H[\bar{D}]$, $T_k \in B(H,H)$ and $T_k = T_k^*$ $(k = 1,2,\ldots,N)$, and let $\{E_k(\lambda)\}$ be a resolution of identities for the operator $T_k (k = 1,2,\ldots,N)$, then we have

$$f(T_1,T_2,\ldots,T_N) = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\xi_1,\xi_2,\ldots,\xi_N)\cdot$$

$$\cdot R_{\xi_1}(T_1)R_{\xi_2}(T_2)\cdots R_{\xi_N}(T_N)\,d\xi_1\cdot d\xi_2\cdots d\xi_N$$

$$= \int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2}\cdots\int_{\alpha_N}^{\beta_N} f(\lambda_1,\lambda_2,\ldots,\lambda_N)\,dE_1(\lambda)\cdot dE_2(\lambda)\cdots dE_N(\lambda)$$

where

$$\alpha_k = \min(C_k \cap R), \qquad \beta_k = \max(C_k \cap R) \qquad (k = 1,2,\ldots,N)$$

and R is a real axis.

Proof. The resolvent operator $R_{\xi_k}(T_k)$ can be represented in the form

$$R_{\xi_k}(T_k) = \frac{1}{\xi_k - T_k} = \sum_{n_k=0} T_k^{n_k}\xi_k^{-n_k-1} \qquad (k = 1,2,\ldots N).$$

The right-hand side series is uniformly convergent for ξ_k since $\xi_k \in C_k$, and $|\xi_k| > \|T_k\|$. Consequently we have

$$R_{\xi_1}(T_1)\cdot R_{\xi_2}(T_2)\ldots R_{\xi_N}(T_N)$$

$$= \Big(\sum_{n_1=0}^{\infty} T^{n_1}\xi_1^{-n_1-1}\Big)\Big(\sum_{n_2=0}^{\infty} T^{n_2}\xi_2^{-n_2-1}\Big)\ldots\Big(\sum_{n_N=0}^{\infty} T_N^{n_N}\xi_N^{-n_N-1}\Big),$$

and

$$\frac{1}{2\pi i}\int_{C_1} \sum_{n_1=0}^{\infty} T_1^{n_1}\xi_1^{-n_1-1} f(\xi_1,\xi_2,\ldots,\xi_N)\,d\xi_1$$

$$= \frac{1}{2\pi i}\int_{C_1} \sum_{n_1=0}^{\infty} T_1^{n_1} \frac{f(\xi_1,\xi_2,\ldots,\xi_N)}{(\xi_1-0)^{n_1+1}}\,d\xi_1$$

$$= \sum_{n_1=0}^{\infty} T_1^{n_1} \frac{1}{2\pi i}\int_{C_1} \frac{f(\xi_1,\xi_2,\ldots,\xi_N)}{(\xi_1-0)^{n_1+1}}\,d\xi_1$$

$$= \sum_{n_1=0}^{\infty} T_1^{n_1} \frac{1}{n_1!} \Big[\frac{\partial^{n_1} f(\xi_1,\xi_2,\ldots\xi_N)}{\partial \xi_1^{n_1}}\Big]_{\xi_1=0}.$$

Therefore, we have

$$f(T_1,T_2,\ldots,T_N) = \Big(\sum_{n_1=0}^{\infty} T_1^{n_1}\frac{1}{n_1!}\Big)\Big(\sum_{n_2=0}^{\infty} T_2^{n_2}\frac{1}{n_2!}\Big)\ldots$$

$$\ldots \sum_{n_N=0}^{\infty} T_N^{n_N}\frac{1}{n_N!}\Big(\Big[\frac{\partial^{n_1+n_2+\ldots+n_N}}{\partial\xi_1^{n_1}\partial\xi_2^{n_2}\ldots\partial\xi_N^{n_N}} f(\xi_1,\xi_2,\ldots,\xi_N)\Big]_{\xi_1=0,\ldots,\xi_N=0}\Big)$$

$$\therefore\ f(T_1,T_2,\ldots,T_N) = \Big[\Big(\sum_{n_1=0}^{\infty} T_1^{n_1}\frac{1}{n_1!}\Big)\Big(\sum_{n_2=0}^{\infty} T_2^{n_2}\frac{1}{n_2!}\Big)\ldots\Big(\sum_{n_N=0} T_N^{n_N}\frac{1}{n_N!}\Big)\Big]$$

$$\times\ \left[\frac{\partial^{n_1+n_2+\cdots+n_N}}{\partial\xi_1^{n_1}\partial\xi_2^{n_2}\cdots\partial\xi_N^{n_N}} f(\xi_1,\xi_2,\ldots,\xi_N)\right]_{\xi_1=0,\xi_2=0,\ldots,\xi_N=0}.$$

Substituting the well-known equalities

$$T = \int \lambda dE_\lambda \qquad \text{and} \qquad T^n = \int \lambda^n dE_\lambda$$

we have

$$\sum_{n_k=0}^{\infty} T_k^{n_k} \frac{1}{n_k!} = \sum_{n_k=0}^{\infty} \int_{\alpha_k}^{\beta_k} \frac{1}{n_k!} \lambda_k^{n_k} dE_k(\lambda) = \int_{\alpha_k}^{\beta_k} \sum_{n_k=0}^{\infty} \frac{1}{n_k!} \lambda_k^{n_k} dE_k(\lambda)$$

and

$$f(T_1,T_2,\ldots,T_N) = \left(\int_{\alpha_1}^{\beta_1} \sum_{n_1=0}^{\infty} \frac{1}{n_1!} \lambda_1^{n_1} dE_1(\lambda) \left[\frac{\partial^{n_1}}{\partial\xi_1^{n_1}}\right]_{\xi_1=0}\right)\cdots$$

$$\cdots\left(\int_{\alpha_N}^{\beta_N} \sum_{n_N=0}^{\infty} \frac{1}{n_N!} \lambda_N^{n_N} dE_N(\lambda) \left[\frac{\partial^{n_N}}{\partial\xi^{n_N}}\right]_{\xi_N=0}\right)\cdot f(\xi_1,\xi_2,\ldots\xi_N).$$

This is allowed by uniform continuity and by the fact that $f(\xi_1,\xi_2,\ldots,\xi_N)$ is a holomorphic function on $\bar{D}(f) \supset \prod_{k=1}^{N} \sigma(T_k)$. Taking the holomorphic function $f(\xi_1,\xi_2,\ldots,\xi_N)$ through the first parenthesis on through the last parenthesis, we have

$$f(T_1,T_2,\ldots,T_N) = \left(\int_{\alpha_1}^{\beta_1} f(\lambda_1,\xi_2,\xi_3,\ldots,\xi_N) dE_1(\lambda)\right)$$

$$\times\left(\int_{\alpha_2}^{\beta_2} \sum_{n_2=0}^{\infty} \frac{1}{n_2!} \lambda^{n_2} dE_2(\lambda) \left[\frac{\partial^{n_2}}{\partial\xi_2^{n_2}}\right]_{\xi_2=0}\right)\cdots\left(\int_{\alpha_N}^{\beta_N} \sum_{n_N=0}^{\infty} \frac{1}{n_N!} \lambda_N^{n_N} dE_N(\lambda) \left[\frac{\partial^{n_N}}{\partial\xi_N^{n_N}}\right]_{\xi_N=0}\right)$$

$$= \left(\int_{\alpha_1}^{\beta_1}\int_{\alpha_2}^{\beta_2} f(\lambda_1,\lambda_2,\xi_3,\xi_4,\ldots,\xi_N) dE_1(\lambda) dE_2(\lambda)\right)$$

$$\times\left(\int_{\alpha_3}^{\beta_3} \sum_{n_3=0}^{\infty} \frac{1}{n_3!} \lambda_3^{n_3} dE_3(\lambda) \left[\frac{\partial^{n_3}}{\partial\xi_3^{n_3}}\right]_{\xi_3=0}\right)$$

$$\cdots\left(\int_{\alpha_N}^{\beta_N} \sum_{n_N=0}^{\infty} \frac{1}{n_N!} \lambda_N^{n_N} dE_N(\lambda) \left[\frac{\partial^{n_N}}{\partial\xi_N^{n_N}}\right]_{\xi_N=0}\right)$$

$$= \ldots$$

$$= \int_{\alpha_1}^{\beta_1} dE_1(\lambda) \int_{\alpha_2}^{\beta_2} dE_2(\lambda) \ldots \int_{\alpha_N}^{\beta_N} f(\lambda_1, \lambda_2, \ldots, \lambda_N)\, dE_N(\lambda);$$

that is,

$$f(T_1, T_2, \ldots, T_N) = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \ldots \int_{\alpha_N}^{\beta_N} f(\lambda_1, \lambda_2, \ldots, \lambda_N)\, dE_1(\lambda)\, dE_2(\lambda) \ldots dE_N(\lambda),$$

where

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) \in \prod_{k=1}^{N} [m_k, M_k]$$

and

$$\prod_{k=1}^{N} [m_k, M_k] \supseteq \prod_{k=1}^{N} \sigma(T_k).$$

This completes the proof.

The reduced formula is obviously a generalization of the form

$$f(T) = \int_{\alpha}^{\beta} f(\lambda)\, dE(\lambda)$$

and this can be deduced from the integral

$$f(T) = \frac{1}{2\pi i} \int_C f(\zeta) R_\zeta(T)\, d\zeta$$

as described in proposition 3.

Using Theorem 1, we have the following:

Theorem 3. Let $f(\xi_1, \xi_2, \ldots, \xi_N) \in H[\bar{D}]$ be a non-vanishing function on $\bar{D}(f)$, then the equality

$$\sigma[f(T_1, T_2, \ldots, T_N)^{-1}] = [\sigma(f(T_1, T_2, \ldots, T_N))]^{-1}$$

is valid.

Proof. Since $f(\xi_1, \xi_2, \ldots, \xi_N) \neq 0$ on $\bar{D}(f)$, the inverse function

$$f(\xi_1, \xi_2, \ldots, \xi_N)^{-1} = g(\xi_1, \xi_2, \ldots, \xi_N)$$

is holomorphic, whence

$$g(\xi_1,\xi_2,\ldots,\xi_N)\in H[\bar{D}].$$

Therefore, according to Lemma 2, this corresponds to an operator

$$g(T_1,T_2,\ldots,T_N) = [f(T_1,T_2,\ldots,T_N)]^{-1}.$$

Using the Generalized Spectral Mapping Theorem, we have the following equalities:

$$\sigma[f(T_1,T_2,\ldots,T_N)^{-1}] = \sigma[g(T_1,T_2,\ldots,T_N)]$$

$$= g(\sigma(T_1) \times \sigma(T_2)\times \ldots \times\sigma(T_N))$$

$$= [f(\sigma(T_1) \times \sigma(T_2)\times \ldots \times\sigma(T_N))]^{-1}$$

$$= [\sigma(f(T_1,T_2,\ldots,T_N))]^{-1}.$$

In the proof of this theorem, we have the following immediate results.

Corollary.

$$\sigma\left([f(T_1,T_2,\ldots,T_N)]^{-1}\right) = \left\{f(\lambda_1,\lambda_2,\ldots,\lambda_N)^{-1}:(\lambda_1,\lambda_2,\ldots,\lambda_N) \in \prod_{k=1}^{N} \sigma(T_k)\right\}.$$

Proof. Since $[f(T_1,T_2,\ldots,T_N)^{-1}] = [f(\sigma(T_1) \times \sigma(T_2)\times \ldots \times\sigma(T_N))]^{-1}$

by the proof of the above theorem, we get the desired result.

2. Spectra of Composition Operators.

We define respectively the addition, multiplication and scalar multiplication of two operators

$$f(T_1,T_2,\ldots,T_N), \quad g(T_1,T_2,\ldots,T_N)\in \quad A[T_1,T_2,\ldots,T_N]$$

by

$$(f+g)(T_1,T_2,\ldots,T_N) = f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)$$

$$(f\cdot g)(T_1,T_2,\ldots,T_N) = f(T_1,T_2,\ldots,T_N)\cdot g(T_1,T_2,\ldots,T_N)$$

and

$$(\alpha f)(T_1,T_2,\ldots,T_N) = \alpha f(T_1,T_2,\ldots,T_N), \quad \alpha\in\mathbb{C}-\{0\}.$$

For the spectra of these operators, we have the following.

Theorem 4.

$$\begin{aligned} &\sigma[f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)] \\ &= \sigma[f(T_1,T_2,\ldots,T_N)]+\sigma[g(T_1,T_2,\ldots,T_N)] \end{aligned} \tag{2-1}$$

$$\begin{aligned} &\sigma[f(T_1,T_2,\ldots,T_N)\cdot g(T_1,T_2,\ldots,T_N)] \\ &\subseteq \sigma[f(T_1,T_2,\ldots,T_N)]\cdot\sigma[q(T_1,T_2,\ldots,T_N)] \end{aligned} \tag{2-2}$$

and

$$\sigma[(\alpha f)(T_1,T_2,\ldots,T_N)] = \alpha\cdot\sigma[f(T_1,T_2,\ldots,T_N)] \tag{2-3}$$

where

$$f(T_1,T_2,\ldots,T_N), \quad g(T_1,T_2,\ldots,T_N)\in \quad A[T_1,T_2,\ldots,T_N]$$

$$\alpha\in\mathbb{C}-\{0\}, \quad \text{and} \quad f \neq g.$$

Proof. It is easily seen that

$$f(\zeta_1,\zeta_2,\ldots,\zeta_N)+g(\zeta_1,\zeta_2,\ldots,\zeta_N)\in H[\bar{D}]$$

$$f(\zeta_1,\zeta_2,\ldots,\zeta_N)\cdot g(\zeta_1,\zeta_2,\ldots,\zeta_N)\in H[\bar{D}]$$

and

$$\alpha f(\zeta_1,\zeta_2,\ldots,\zeta_N)\in H[\bar{D}].$$

According to Lemma 2, these correspond to the above operators. By Theorem 1,

$$[f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)] = \sigma[(f+g)(T_1,T_2,\ldots,T_N)$$

$$= (f+g)(\sigma(T_1) \times \sigma(T_2)\times \ldots \times\sigma(T_N))$$

$$= \{f(\lambda_1,\lambda_2,\ldots,\lambda_N)+g(\lambda_1,\lambda_2,\ldots,\lambda_N)\,|\,(\lambda_1,\lambda_2,\ldots,\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)\}$$

$$\subseteq f(\sigma(T_1) \times \sigma(T_2)\times \ldots \times\sigma(T_N))+g(\sigma(T_1) \times \sigma(T_2)\times \ldots \times\sigma(T_N))$$

$$= \sigma[f(T_1,T_2,\ldots,T_N)]+\sigma[g(T_1,T_2,\ldots,T_N)].$$

For the converse inclusion, we need some simple calculations.

Taking any $f(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\sigma[f(T_1,T_2,\ldots,T_N)]$ and $g(\mu_1,\mu_2,\ldots,\mu_N)$ $\in \sigma[g(T_1,T_2,\ldots,T_N)]$,

$$f(\lambda_1,\lambda_2,\ldots,\lambda_N)+g(\mu_1,\mu_2,\ldots,\mu_N)\in\sigma[f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)];$$

for if

$$f(\lambda_1,\lambda_2,\ldots,\lambda_N)+g(\mu_1,\mu_2,\ldots,\mu_N)\bar{\in}\sigma[f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)],$$

then there exists an inverse operator

$$[f(\lambda_1,\lambda_2,\ldots,\lambda_N)+g(\mu_1,\mu_2,\ldots,\mu_N)-\{f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)\}]^{-1}$$

$$= \frac{1}{[f(\lambda_1,\lambda_2,\ldots,\lambda_N)-f(T_1,T_2,\ldots,T_N)]\{1+\frac{g(\mu_1,\mu_2,\ldots,\mu_N)-g(T_1,T_2,\ldots T_N)}{f(\lambda_1,\lambda_2,\ldots,\lambda_N)-f(T_1,T_2,\ldots T_N)}\}}.$$

This contradicts the facts that

$$f(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\sigma[f(T_1,T_2,\ldots,T_N)]$$

$$g(\mu_1,\mu_2,\ldots,\mu_N)\in\sigma[g(T_1,T_2,\ldots,T_N)],$$

whence

$$\sigma[f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)] \subseteq \sigma[f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)].$$

Thus we have (2-1).

$$\sigma[f(T_1,T_2,\ldots,T_N)\cdot g(T_1,T_2,\ldots,T_N)] = \sigma[(f\cdot g)(T_1,T_2,\ldots,T_N)]$$

$$= (f\cdot g)(\sigma(T_1) \times \sigma(T_2)\times \ldots \times\sigma(T_N))$$

$$= \{f(\lambda_1,\lambda_2,\ldots,\lambda_N)g(\lambda_1,\lambda_2,\ldots,\lambda_N) \mid (\lambda_1,\lambda_2,\ldots,\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)\}$$

$$\subseteq \{f(\lambda_1,\lambda_2,\ldots,\lambda_N) \mid (\lambda_1,\lambda_2,\ldots,\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)\}$$

$$\times \{g(\lambda_1,\lambda_2,\ldots,\lambda_N) \mid (\lambda_1,\lambda_2,\ldots,\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)\}$$

$$= [f(T_1,T_2,\ldots,T_N)]\cdot\sigma[g(T_1,T_2,\ldots,T_N)];$$

i.e., $[f(T_1,T_2,\ldots,T_N)\cdot g(T_1,T_2,\ldots,T_N)]$

$$\subseteq \sigma[f(T_1,T_2,\ldots,T_N)]\cdot\sigma[g(T_1,T_2,\ldots,T_N)].$$

The converse relation does not hold in general, but if

$$f(T_1,T_2,\ldots,T_N) = g(T_1,T_2,\ldots,T_N), \text{ then equality}$$

will be held. This will be proved later.

Finally, it is obvious that

$$\sigma[\alpha\cdot f(T_1,T_2,\ldots,T_N)] = \alpha\cdot\sigma[f(T_1,T_2,\ldots,T_N)] \qquad (\alpha \neq 0),$$

since

$$(\alpha f)(\lambda_1,\lambda_2,\ldots,\lambda_N) = \alpha\cdot f(\lambda_1,\lambda_2,\ldots,\lambda_N).$$

Now we propose the following equality:

$$\sigma[f(T_1,T_2,\ldots,T_N)^2] = \{\sigma[f(T_1,T_2,\ldots,T_N)]\}^2. \tag{2-4}$$

This is a special case of (2-2). The inclusion

$$\sigma[f(T_1,T_2,\ldots,T_N)^2] \subseteq \{\sigma[f(T_1,T_2,\ldots,T_N)]\}^2$$

is obvious from the proof of Theorem 4. Hence all we have to do is to prove converse inclusion

$$\sigma[f(T_1,T_2,\ldots,T_N)^2] \supseteq \{\sigma[f(T_1,T_2,\ldots,T_N)]\}^2.$$

For any

$$f(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\sigma[f(T_1,T_2,\ldots,T_N)],$$

$$f(\lambda_1,\lambda_2,\ldots,\lambda_N)^2-f(T_1,T_2,\ldots,T_N)^2$$

$$= \{f(\lambda_1,\lambda_2,\ldots,\lambda_N)+f(T_1,T_2,\ldots,T_N)\}\{f(\lambda_1,\lambda_2,\ldots,\lambda_N)-f(T_1,T_2,\ldots,T_N)\}$$

$$\therefore\ f(\lambda_1,\lambda_2,\ldots,\lambda_N)^2 \in \sigma[f(T_1,T_2,\ldots,T_N)^2].$$

That is

$$\{\sigma[f(T_1,T_2,\ldots,T_N)]\}^2 \subseteq \sigma[f(T_1,T_2,\ldots,T_N)^2].$$

Moreover, we have the equality

$$\sigma[f(T_1,T_2,\ldots,T_N)^n] = \{\sigma[f(T_1,T_2,\ldots,T_N)]\}^n, \tag{2-5}$$

for any positive integer n. This follows immediately by induction. Furthermore, we obtain the following result.

Theorem 5. Let p(z) be any polynomial which does not contain a constant term, then we have the equality

$$\sigma[p\{f(T_1,T_2,\ldots,T_N)\}] = p\{\sigma[f(T_1,T_2,\ldots,T_N)]\}. \tag{2-6}$$

Proof. Let $p(z) = a_o z^n + a_1 z^{n-1} + \ldots + a_{n-1} z$.

$$\sigma[p\{f(T_1,T_2,\ldots,T_N)\}] = \sigma[a_0 f(T_1,T_2,\ldots T_N)^n$$

$$+ a_1 f(T_1,T_2,\ldots T_N)^{n-1} + \ldots + a_{n-1} f(T_1,T_2,\ldots T_N)]$$

$$= a_0\sigma[f(T_1,T_2,\ldots,T_N)^n] + a_1\cdot\sigma[f(T_1,T_2,\ldots,T_N)^{n-1} + \ldots + a_{n-1}\sigma[f(T_1,T_2,\ldots,T_N)]$$

$$= a_0\{\sigma[f(T_1,T_2,\ldots T_N)]\}^n + a_1\{\sigma[f(T_1,T_2,\ldots,T_N)]\}^{n-1} + \ldots + a_{n-1}$$

$$\times\ \sigma[f(T_1,T_2,\ldots,T_N)]$$

$$= p\{\sigma[f(T_1,T_2,\ldots,T_N)]\}.$$

This proof employs (2-1), (2-3) and (2-5). This theorem shows that σ can be commuted by p, and the following question naturally arises: Can it be that

$$\sigma[g(f(T_1,T_2,\ldots T_N))] = g(\sigma[f(T_1,T_2,\ldots,T_N)])$$

for any holomorphic function g(z)?

This question is answered in the affirmative. We prove this directly in the following way.

For a non-constant function $f(\xi_1,\xi_2,\ldots,\xi_N)\in H[\bar{D}]$ whose range is R(f), we consider the second function g which is holomorphic on $D_1(g) \supseteq R(f)$. Then $g[f(\xi_1,\xi_2,\ldots,\xi_N)]$ is holomorphic on $\bar{D}(f)$, that is

$$g[f(\xi_1,\xi_2,\ldots,\xi_N)]\in H[\bar{D}].$$

Therefore this corresponds to an operator

$$g[f(T_1,T_2,\ldots,T_N)] = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} g[f(\xi_1,\xi_2,\ldots,\xi_N)]$$

$$\times R_{\xi_1}(T_1)\ldots R_{\xi_N}(T_N)\,d\xi_1 d\xi_2\ldots d\xi_N.$$

Putting

$$g[f(\xi_1,\xi_2,\ldots,\xi_N)] = F(\xi_1,\xi_2,\ldots,\xi_N),$$

we have

$$g[f(T_1,T_2,\ldots,T_N)] = F(T_1,T_2,\ldots,T_N).$$

Hence

$$\sigma\,[g(f(T_1,T_2,\ldots,T_N))] = \sigma[F(T_1,T_2,\ldots,T_N)]$$

$$= F(\sigma(T_1)\times\sigma(T_2)\times\ \ldots\ \times\sigma(T_N))$$

$$= g[f(\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_N))]$$

$$= g(\sigma[f(T_1, T_2, \ldots, T_N)]).$$

We have used the Generalized Spectral Mapping Theorem to obtain the above equality. Therefore we have the following:

Theorem 6. For a holomorphic function g on a domain $D_1(g) \supseteq R(f)$ and $f(\xi_1, \xi_2, \ldots, \xi_N) \in H[\bar{D}]$, we have

$$\sigma[g(f(T_1, T_2, \ldots, T_N))] = g(\sigma[f(T_1, T_2, \ldots, T_N)]).$$

Corollary. For

$$f(T) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - T}\, d\zeta,$$

we have

$$\sigma[g(f(T))] = g(\sigma[f(T)]) = g(f(\sigma(T))).$$

In Chapter II, (vii), we saw that

$$f^{(n)}(T) = \frac{1}{2\pi i}\int_C \{\frac{d^n}{d\zeta^n} f(\zeta)\} R_\zeta(T)\, d\zeta$$

$$= \frac{n!}{2\pi i}\int_C f(\zeta) \cdot R_\zeta(T)^{n+1} d\zeta.$$

We shall discuss this formula for an operator

$$f(T_1, T_2, \ldots, T_N) = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2} \cdots \int_{C_N} f(\zeta_1, \zeta_2, \ldots, \zeta_N)$$

$$\times R_{\zeta_1}(T_1) R_{\zeta_2}(T_2) \cdots R_{\zeta_N}(T_N)\, d\zeta_1 d\zeta_2 \cdots d\zeta_N.$$

If $f(\zeta_1, \zeta_2, \ldots, \zeta_N)$ is holomorphic on $\bar{D}(f)$, then the partial differential

$$\frac{\partial^{n_k} f(\zeta_1,\zeta_2,\ldots,\zeta_N)}{\partial\zeta_k^{n_k}}$$

is also holomorphic on $\bar{D}(f)$ for each k. We put

$$f^{(n_k)}(\zeta_1,\zeta_2,\ldots,\zeta_{n_k},\ldots\zeta_N) = \frac{\partial^{n_k} f(\zeta_1,\zeta_2,\ldots,\zeta_N)}{\partial\zeta_k^{n_k}}, \quad n_k = (0,0,\ldots,n_k,0,\ldots 0).$$

This then corresponds to an operator

$$f^{(n_k)}(T_1,T_2,\ldots,T_N).$$

We define

$$n = (n_1,n_2,\ldots,n_N), \qquad |n| = n_1+n_2+\ldots+n_N$$

and

$$f^{(n)}(\zeta_1,\zeta_2,\ldots,\zeta_N) = \frac{\partial^{|n|} f(\zeta_1,\zeta_2,\ldots,\zeta_N)}{\partial\zeta_1^{n_1}\partial\zeta_2^{n_2}\ldots\partial\zeta_N^{n_N}}.$$

By the same argument, this corresponds to an operator

$$f^{(n)}(T_1,T_2,\ldots,T_N)$$

such that

$$f^{(n)}(T_1,T_2,\ldots,T_N) = \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} f^{(n)}(\xi_1,\xi_2,\ldots,\xi_N)$$

$$\times R_{\zeta_1}(T_1)\ldots R_{\zeta_N}(T_N)d\zeta_1 d\zeta_2\ldots d\zeta_N;$$

and it is easily seen that

$$f^{(n)}(T_1,T_2,\ldots,T_N) = \frac{n_1!n_2!\ldots n_N!}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\zeta_1,\zeta_2,\ldots,\zeta_N)$$

$$\times R_{\zeta_1}(T_1)^{n_1+1}\ldots R_{\zeta_N}(T_N)^{n_N+1}d\zeta_1 d\zeta_2\ldots d\zeta_N$$

$$= \frac{1}{(2\pi i)^N} \frac{\partial^{|n|}}{\partial T_1^{n_1} \partial T_2^{n_2} \cdots \partial T_N^{n_N}} \int_{C_1} \int_{C_2} \cdots \int_{C_N} f(\zeta_1, \zeta_2, \ldots, \zeta_N)$$

$$\times R_{\zeta_1}(T_1) \cdots R_{\zeta_N}(T_N) d\zeta_1 d\zeta_2 \cdots d\zeta_N$$

$$= \frac{\partial^{|n|} f(T_1, T_2, \ldots, T_N)}{\partial T_1^{n_1} \partial T_2^{n_2} \cdots \partial T_N^{n_N}},$$

that is,

$$f^{(n)}(T_1, T_2, \ldots, T_N) = \frac{\partial^{|n|} f(T_1, T_2, \ldots, T_N)}{\partial T_1^{n_1} \partial T_2^{n_2} \cdots \partial T_N^{n_N}}$$

$$= \frac{1}{(2\pi i)^N} \int_{C_1} \int_{C_2} \cdots \int_{C_N} f^{(n)}(\zeta_1, \zeta_2, \ldots, \zeta_N) R_{\zeta_1}(T_1) R_{\zeta_2}(T_2) \cdots R_{\zeta_N}(T_N) d\zeta_1 \cdots d\zeta_N.$$

Hence we have the following:

Theorem 7. If $f(\zeta_1, \zeta_2, \ldots, \zeta_N)$ is a holomorphic function on $\bar{D}(f)$, the equality

$$\sigma\left[\frac{\partial^{|n|} f(T_1, T_2, \ldots, T_N)}{\partial T_1^{n_1} \partial T_2^{n_2} \cdots \partial T_N^{n_N}}\right] = \left\{\frac{\partial^{|n|} f(\lambda_1, \lambda_2, \ldots, \lambda_N)}{\partial \lambda_1^{n_1} \partial \lambda_2^{n_2} \cdots \partial \lambda_N^{n_N}} : (\lambda_1, \lambda_2, \ldots, \lambda_N) \in \prod_{k=1}^{N} \sigma(T_k)\right\}$$

holds, where

$$n = (n_1, n_2, \ldots, n_N), \quad |n| = n_1 + n_2 + \cdots + n_N.^{\dagger}$$

Proof. We note that a holomorphic function $f(z)$ guarantees the existence of $f^{(n)}(z)$ for any natural number

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Also, a holomorphic function $f(\zeta_1, \zeta_2, \ldots, \zeta_N)$ on $\bar{D}(f)$ assures us the

† See Reference 8.

existence of $f^{(n)}(\zeta_1,\zeta_n,\ldots,\zeta_N)$ for $n = (n_1,n_2,\ldots,n_N)$. This is defined above. Since

$$\sigma\left[\frac{\partial^{|n|} f(T_1,T_2,\ldots,T_N)}{\partial T_1^{n_1}\partial T_2^{n_2}\ldots\partial T_N^{n_N}}\right] = \sigma[f^{(n)}(T_1,T_2,\ldots,T_N)]$$

$$= f^{(n)}(\sigma(T_1)\times\sigma(T_2)\times\ldots\times\sigma(T_N))$$

$$= \left\{\frac{\partial^{|n|} f(\lambda_1,\lambda_2,\ldots,\lambda_N)}{\partial\lambda_1^{n_1}\partial\lambda_2^{n_2}\ldots\partial\lambda_N^{n_N}} : (\lambda_1,\lambda_2,\ldots,\lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)\right\},$$

we have proved the theorem.

Putting

$$\left\{\frac{\partial^{|n|} f(\lambda_1,\lambda_2,\ldots,\lambda_N)}{\partial\lambda_1^{n_1}\partial\lambda_2^{n_2}\ldots\partial\lambda_N^{n_N}} : (\lambda_1,\lambda_2,\ldots,\lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)\right\}$$
$$= \frac{\partial^{|n|} f(\sigma(T_1)\times\ldots\times\sigma(T_N))}{\partial\lambda_1^{n_1}\partial\lambda_2^{n_2}\ldots\partial\lambda_N^{n_N}},$$

the result of Theorem 7 can be written in the form

$$\sigma\left[\frac{\partial^{|n|} f(T_1,T_2,\ldots,T_N)}{\partial T_1^{n_1}\partial T_2^{n_2}\ldots\partial T_N^{n_N}}\right] = \frac{\partial^{|n|} f(\sigma(T_1)\times\sigma(T_2)\times\ldots\times\sigma(T_N))}{\partial\lambda_1^{n_1}\partial\lambda_2^{n_2}\ldots\partial\lambda_N^{n_N}}$$

or

$$\sigma[D_T^{(n)} f(T_1,T_2,\ldots,T_N)] = D_\lambda^{(n)}\sigma[f(T_1,T_2,\ldots,T_N)].$$

Hence we have the following.

Corollary. If $f(\zeta_1,\zeta_2,\ldots,\zeta_N)\in H[\bar{D}]$, then

$$\sigma[D_T^{(n)} f(T_1,T_2,\ldots,T_N)] = D_\lambda^{(n)}\sigma[f(T_1,T_2,\ldots,T_N)]$$

where

$$D_T^{(n)} = \frac{\partial^{|n|}}{\partial T_1^{n_1}\partial T_2^{n_2}\ldots\partial T^{n_N}} \quad\text{and}\quad D_\lambda^{(n)} = \frac{\partial^{|n|}}{\partial\lambda_1^{n_1}\partial\lambda_2^{n_2}\ldots\partial\lambda_N^{n_N}}.$$

Remark. We now have a holomorphic function f(z) on D(f) guaranteeing the existence of n-th derivatives $f^{(n)}(z)$ for any positive integer n. This is seen by observing the following facts.

Any holomorphic function f(z) on D(f) can be represented by

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{(Cauchy Integral Formula)}$$

and

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \qquad \text{since } f(z) \text{ is holomorphic.}$$

This also requires the existence of

$$f''(z) = \frac{2}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta$$

and so on; $f^{(n-1)}(z)$ guarantees the existence of

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

This obtains the asserted statement.

In the above discussion, we assumed that only f(z) is holomorphic, but since f"(z) exists by f'(z), f'(z) is automatically holomorphic and so on. Hence without assuming that f'(z),f"(z),... are holomorphic, they automatically come out to be holomorphic.

3. Strong Operator Topology in $A[T_1, T_2, \ldots, T_N]$.

In this section we shall induce a topology on the set $A[T_1, T_2, \ldots, T_N]$. The Banach space B(X,X) of continuous linear mappings $T: X \to X$ has three most commonly used topologies, namely the uniform, strong and weak operator topologies, as given in the following definitions.

Definition 3-1. The uniform operator topology in B(X,X) is the metric topology of B(X,X) induced by its norm

$$||T|| = \sup_{||x|| \le 1} ||Tx||.$$

Definition 3-2. The strong operator topology in $B(X,X)$ is the topology given by the basic set of neighborhoods

$$U(T:F,\varepsilon) = \{S \,|\, S \in B(X,X), ||(T-S)x|| < \varepsilon, x \in F\},$$

where F is an arbitrary finite subset of X, and $\varepsilon > 0$ is arbitrary. By definition, a generalized sequence $\{T_\alpha\}$ converges to T if and only if $\{T_\alpha x\}$ converges to Tx for every x in X.

Definition 3-3. The weak operator topology in $B(X,X)$ is the topology defined by the basic set of neighborhoods

$$U(T,F^*,A,\varepsilon) = \{S \,|\, S\ B(X,X), |x^*(T-S)x| < \varepsilon, x^* \in F^*, x \in A\},$$

where F^* and A are arbitrary finite subsets of elements in X^* and X, X^* is the dual space of X. Thus, in the weak topology, a generalized sequence $\{T_\alpha\}$ converges to T if and only if $\{x^* T_\alpha x\}$ converges to $x^* Tx$ for every x in X and x^* in X^*.

It is evident that the uniform operator topology is stronger than the strong operator topology, and that the strong operator topology is stronger than the weak operator topology. With its uniform operator topology, $B(X,X)$ will be a Banach space.

(a) If X is a Banach space, then $Y = X \times X \times \ldots \times X$, the set of all N-tuples $y = (x_1,x_2,\ldots,x_N)$ with $x_i \in X (i = 1,2,\ldots,N)$, is also Banach space with norm: $||(x_1,x_2,\ldots,x_N)|| = \sup_{1 \le k \le N} ||x_k||$.

Proof. Let $(x_1,x_2,\ldots,x_N)$, $(x_1',x_2',\ldots,x_N')$ be two elements in Y. By defining

$$(X_1,X_2,\ldots,X_N)+(x_1',x_2',\ldots,x_N') = (x_1+x_1',\ldots,x_N+x_N')$$

and

$$\alpha(x_1,x_2,\ldots,x_N) = (\alpha x_1,\alpha x_2,\ldots,\alpha x_N),$$

Y forms a linear space. Since

$$\|(x_1,x_2,\ldots,x_N)\| \geq 0$$

$$\|\alpha(x_1,x_2,\ldots,x_N)\| = \sup_{1\leq k\leq N}\|\alpha x_i\| = |\alpha|\sup_{1\leq k\leq N}\|x_k\|$$

$$= |\alpha|\cdot\|(x_1,x_2,\ldots,x_N)\|$$

and

$$\|(X_1,X_2,\ldots,X_n)+(x_1',x_2',\ldots,x_N')\| = \sup_{1\leq k\leq N}\|x_k+x_k'\| \leq \sup_{1\leq k\leq N}\|x_k\|+\sup_{1\leq k\leq N}\|x_N'\|$$

$$= \|(x_1,x_2,\ldots,x_N)\|+\|(x_1',x_2',\ldots,x_N')\|.$$

This means that Y is a normed linear space, so all we have to do is to prove the completeness of Y. This is easily proved in the following way.

For any Cauchy sequency $\{(x_1^{(n)},x_2^{(n)},\ldots,x_N^{(n)}):x_k^{(n)}\in X \text{ for each } k\}$, $\|(x_1^{(n)},x_2^{(n)},\ldots,x_N^{(n)})-(x_1^{(m)},x_2^{(m)},\ldots,x_N^{(m)})\| = \sup_{1\leq k\leq N}\|x_k^{(n)}-x_k^{(m)}\| \to 0$, as $m,n \to \infty$. So, $\{x_k^{(n)}\}_n$ is a Cauchy sequence in X, but X is a Banach space, whence there exists a point $x_k^{(0)}$ in X such that

$$x_k^{(n)} \to x_k^{(0)} \qquad \text{as} \qquad n \to \infty.$$

Therefore, for each $k = 1,2,\ldots N$, $x_k^{(n)} \to x_k^{(0)}$ as $n \to \infty$. Hence

$$(x_1^{(0)},x_2^{(0)},\ldots,x_N^{(0)})\in Y.$$

(b) Let $T_k \in B(X,X)$ $k = 1,2,\ldots,N$. We put the domain of

$$\xi I - T = D_T(\xi)$$

and the range of $\xi I - T = \Delta_T(\xi)$† for any $\xi \in \rho(T)$. Then the domain of $R_\xi(T)$ is $\Delta_T(\xi)$ and the range of $R_\xi(T)$ is $D_T(\xi)$. Hence the resolvent operator $R_\xi(T)$ is a bounded linear operator of $\Delta_T(\xi)$ onto $D_T(\xi)$.

We define

$$\begin{aligned} &R_{\xi_1}(T_1)\cdot R_{\xi_2}(T_2)\ldots R_{\xi_N}(T_N)(x_1,x_2,\ldots,x_N) \\ &= [R_{\xi_1}(T_1)x_1][R_{\xi_2}(T_2)x_2]\ldots[R_{\xi_N}(T_N)x_N] \end{aligned} \tag{3-1}$$

for

$$(x_1,x_2,\ldots,x_N) \in \prod_{k=1}^{N} \Delta_{T_k}(\xi_k).$$

We then have

$$\begin{aligned} &f(T_1,T_2,\ldots,T_N)(x_1,x_2,\ldots,x_N) \\ &= \frac{1}{(2\pi i)^N}\int_{C_1}\int_{C_2}\cdots\int_{C_N} f(\xi_\zeta,\xi_2,\ldots,\xi_N) \\ &\times [R_{\xi_1}(T_1)x_1][R_{\xi_2}(T_2)x_2]\ldots[R_{\xi_N}(T_N)x_N]\cdot d\xi_1\cdot d\xi_2\ldots d\xi_N. \end{aligned} \tag{3-2}$$

Therefore, the operator $f(T_1,T_2,\ldots,T_N)$ can be regarded as a multilinear operator of $\prod_{k=1}^{N} \Delta_{T_k}(\xi_k)$ into the Banach space X. We denote the set of all these operators by

$$A[\prod_{k=1}^{N} \Delta_k(T_k),X].$$

Of course,

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† See Reference 7.

$$A[T_1,T_2,\dots,T_N] = \mathbf{A}[\prod_{k=1}^{N} \Delta_k(T_k),X],$$

but we emphasize the domain and the range.

Now, we define the norm of the operator $f(T_1,T_2,\dots,T_N)$ in the following way:

$$\|f(T_1,T_2,\dots,T_N)\| \tag{3-3}$$

$$= \sup_{\|(x_1,x_2,\dots,x_N)\|\le 1} \|f(T_1,T_2,\dots,T_N)(x_1,x_2,\dots,x_N)\|$$

where

$$\|(x_1,x_2,\dots,x_N)\| = \sup_{1\le k\le N} \|x_k\|$$

as we defined in section 3, (a).

From (3-3) we have the following properties:

$$\|f(T_1,T_2,\dots,T_N)\| \geq 0 \quad (=0 \text{ if and only if } f(T_1,T_2,\dots,T_N) = 0)$$

$$\|\alpha f(T_1,T_2,\dots,T_N)\| = |\alpha| \|f(T_1,T_2,\dots,T_N)\|$$

and

$$\|f(T_1,T_2,\dots,T_N)+g(T_1,T_2,\dots,T_N)\| \leq \|f(T_1,T_2,\dots,T_N)\| + \|g(T_1,T_2,\dots,T_N)\| .$$

The first two equalities are obvious; the triangular inequality is also easily seen by following simple calculation,

$$\|(f(T_1,T_2,\dots,T_N)+g(T_1,T_2,\dots,T_N))(x_1,x_2,\dots x_N)\|$$

$$= \|f(T_1,T_2,\dots,T_N)(x_1,x_2,\dots,x_N)+g(T_1,T_2,\dots T_N)(X_1,X_2,\dots,X_n)\|$$

$$\leq \|f(T_1,T_2,\dots,T_N)(X_1,X_2,\dots,X_N)\| + \|g(T_1,T_2,\dots,T_N)(x_1,x_2,\dots,x_N)\|,$$

since both $f(T_1,T_2,\ldots,T_N)(x_1,x_2,\ldots,x_N)$, $g(T_1,T_2,\ldots,T_N)(x_1,x_2,\ldots,x_N)$ belong to space X. Hence taking the supremum of both sides, we have the desired inequality.

Now, we introduce the Strong Operator Topology.

Definition 3-3. By the strong operator topology in $A[T_1,T_2,\ldots T_N]$ we shall mean a topology induced by the norm $\|\ \|$ which is defined in $A[T_1,T_2,\ldots,T_N]$.

The spherical neighborhood

$$S_\varepsilon(f(T_1,T_2,\ldots,T_N):A_1 \times A_2\times \ldots \times A_N,)$$

$$= \{g(T_1,T_2,\ldots,T_N)\in A[T_1,T_2,\ldots,T_N]:\|(f(T_1,T_2,\ldots,T_N)-g(T_1,T_2,\ldots,T_N))$$

$$(x_1,x_2,\ldots,x_N)\| < \varepsilon,(x_1,x_2,\ldots,x_N)\in \prod_{k=1}^{N} A_k\}$$

where each A_k $(k = 1,2,\ldots,N)$ is a finite set in X. Obviously this basis induces a topology in $A[T_1,T_2,\ldots,T_N]$.

Thus in this operator topology, a sequence $\{f_\alpha(T_1,T_2,\ldots,T_N)\}$ converges to $f_0(T_1,T_2,\ldots,T_N)$ if and only if the sequence $\{f_\alpha(T_1,T_2,\ldots,T_N)(x_1,x_2,\ldots,x_N)$ converges to $f_0(T_1,T_2,\ldots,T_N)$ $(x_1,x_2,\ldots,x_N)$. We are now prepared to discuss the following:

Theorem 8. If $f_n(T_1,T_2,\ldots,T_N)$ converges to $f_0(T_1,T_2,\ldots,T_N)$ with respect to the strong operator topology, then

$$\sigma[f_n(T_1,T_2,\ldots,T_N)] \to \sigma[f_0(T_1,T_2,\ldots,T_N)] \quad \text{as} \quad n \to \infty,$$

with respect to the usual topology in the complex plane.

Proof. We would like to prove that $f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)$ converges uniformly on

$$\sigma(T_1) \times \sigma(T_2) \times \ldots \times \sigma(T_N) = \prod_{k=1}^{N} \sigma(T_k).$$

For any $f_n(\lambda_1,\lambda_2,\ldots,\lambda_N) \in \sigma[f_n(T_1,T_2,\ldots,T_N)]$,

$$f_n(\lambda_1,\lambda_2,\ldots,\lambda_N) - f_m(\lambda_1,\lambda_2,\ldots,\lambda_N) = (f_n - f_m)(\lambda_1,\lambda_2,\ldots,\lambda_N) \qquad (n \neq m)$$

$$= h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N)$$

is an element of

$$\sigma[h_{n,m}(T_1,T_2,\ldots,T_N)].$$

If

$$h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N) \notin \sigma[h_{n,m}(T_1,T_2,\ldots,T_N)],$$

there exists an inverse operator

$$[h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N) - h_{n,m}(T_1,T_2,\ldots,T_N)]^{-1}.$$

But

$$[h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N)I - h_{n,m}(T_1,T_2,\ldots,T_N)]^{-1}$$

$$= \frac{1}{\{f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)I - f_n(T_1,T_2,\ldots,T_N)\}}$$

$$\times \frac{1}{\left\{1 - \dfrac{f_m(\lambda_1,\lambda_2,\ldots,\lambda_N)I - f_m(T_1,T_2,\ldots,T_N)}{f_n(\lambda_1,\lambda_2,\ldots,\ {}_N)I - f_n(T_1,T_2,\ldots,T_N)}\right\}}.$$

This contradicts the fact that

$$f_n(\lambda_1,\lambda_2,\ldots,\lambda_N) \in \sigma[f_n(T_1,T_2,\ldots,T_N)].$$

The existence of $[h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N)-h_{n,m}(T_1,T_2,\ldots,T_N)]^{-1}$ implies that

$$|h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N)| > \|h_{n,m}(T_1,T_2,\ldots,T_N)\|.$$

Thus

$$h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\sigma[h_{n,m}(T_1,T_2,\ldots,T_N)]$$

shows that

$$|h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N)| \le \|h_{n,m}(T_1,T_2,\ldots,T_N)\|.$$

Therefore

$$|h_{n,m}(\lambda_1,\lambda_2,\ldots,\lambda_N)| = |f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)-f_m(\lambda_1,\lambda_2,\ldots,\lambda_N)| \to 0$$

as

$$n,m \to \infty,$$

since

$$\|h_{n,m}(T_1,T_2,\ldots,T_N)\| \to 0,$$

this means that $\{f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)\}$ is a Cauchy sequence in the complex plane, then there is a function $g(\lambda_1,\lambda_2,\ldots,\lambda_N)$ such that

$$f_n(\lambda_1,\lambda_2,\ldots,\lambda_N) \to g(\lambda_1,\lambda_2,\ldots,\lambda_N)$$

as

$$n \to \infty.$$

We have to show that

$$g(\lambda_1,\lambda_2,\ldots,\lambda_N) \equiv f_0(\lambda_1,\lambda_2,\ldots,\lambda_N).$$

For this, let $f_M(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\sigma[f_M(T_1,T_2,\ldots,T_N)]$ for sufficiently large M, so that

$$f_M(T_1,T_2,\ldots,T_N)\in S_{\varepsilon/2}(f_0(T_1,T_2,\ldots,T_N))$$

and

$$|f_M(\lambda_1,\lambda_2,\ldots,\lambda_N)-g(\lambda_1,\lambda_2,\ldots,\lambda_N)| < \varepsilon/2.$$

Suppose that

$$g(\lambda_1,\lambda_2,\ldots,\lambda_N)\bar{\in}\sigma[f_0(T_1,T_2,\ldots,T_N)].$$

Since

$$\|\{f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)I-f_n(T_1,T_2,\ldots,T_N)\}$$

$$-\{g(\lambda_1,\lambda_2,\ldots,\lambda_N)I-f_0(T_1,T_2,\ldots,T_N)\}\|$$

$$\leq |f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)I-g(\lambda_1,\lambda_2,\ldots,\lambda_N)|$$

$$+\|f_n(T_1,T_2,\ldots,T_N)-f_0(T_1,T_2,\ldots,T_N)\|$$

$$< \varepsilon \qquad \text{for} \qquad n \geq M,$$

and

$$[g(\lambda_1,\lambda_2,\ldots,\lambda_N)I-f_0(T_1,T_2,\ldots,T_N)]^{-1}$$

exists, so does

$$[f_M(\lambda_1,\lambda_2,\ldots,\lambda_N)I-f_M(T_1,T_2,\ldots,T_N)]^{-1}.$$

But this contradicts the fact that

$$f_M(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\sigma[f_M(T_1,T_2,\ldots,T_N)];$$

$$\text{i.e.,}\quad g(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\sigma[f_0(T_1,T_2,\ldots,T_N)].$$

According to the Generalized Spectral Mapping Theorem, any number $\mu\in\sigma[f_0(T_1,T_2,\ldots,T_N)]$ can be represented in the form

$$\mu = f_0(\lambda_1,\lambda_2,\ldots,\lambda_N), \qquad (\lambda_1,\lambda_2,\ldots,\lambda_N)\in\prod_{k=1}^{N}\sigma(T_k).$$

Thus we can identify $f_0(\lambda_1,\lambda_2,\ldots,\lambda_N)$ and $g(\lambda_1,\lambda_2,\ldots,\lambda_N)$, that is,

$$f_0(\lambda_1,\lambda_2,\ldots,\lambda_N)\equiv g(\lambda_1,\lambda_2,\ldots,\lambda_N).$$

We have proved the theorem.

Theorem 9. For an operator $f(T_1,T_2,\ldots,T_N)\in A[T_1,T_2,\ldots,T_N]$, there exists a point $(\zeta_1^{(0)},\zeta_2^{(0)},\ldots,\zeta_N^{(0)})\in\bar{D}(f)$, such that

$$\|f(T_1,T_2,\ldots,T_N)\| = |f(\zeta_1^{(0)},\zeta_2^{(0)},\ldots,\zeta_N^{(0)})|.$$

Proof.

$$\|f(T_1,T_2,\ldots,T_N)\| = \sup_{\|(x_1,x_2,\ldots,x_N)\|\le 1}\|f(T_1,T_2,\ldots,T_N)(x_1,x_2,\ldots,x_N)\|$$

$$\le \sup_{\|(X_1,X_2,\ldots,x_N)\|\le 1,(\zeta_1,\zeta_2,\ldots,\zeta_N)\in\bar{D}(f)}|f(\zeta_1,\zeta_2,\ldots,\zeta_N)|$$

$$\times\left\|\frac{1}{2\pi i}\int_{C_1}R_{\zeta_1}(T_1)x_1d\zeta_1\right\|\cdots\left\|\frac{1}{2\pi i}\int_{C_N}R_{\zeta_N}(T_N)x_Nd\zeta_N\right\|$$

$$\le \sup_{(\zeta_1,\zeta_2,\ldots,\zeta_N)\in\bar{D}(f)}|f(\zeta_1,\zeta_2,\ldots,\zeta_N)|\cdot\sup_{1\le k\le N}\|x_k\|\sup_{1\le k\le N}\|x_k\|\cdots\sup_{1\le k\le N}\|x_k\|$$

$$\le \sup_{(\zeta_1,\zeta_2,\ldots,\zeta_N)\in\bar{D}(f)}|f(\zeta_1,\zeta_2,\ldots,\zeta_N)|$$

since

$$\frac{1}{2\pi i}\int_C R_\zeta(T)\,d\zeta = I \text{ (Identity operator)}$$

and

$$\sup_{1\le k\le N} \|x_k\| \le 1,$$

i.e., $\|f(T_1,T_2,\ldots,T_N)\| \le \sup_{(\zeta_1,\zeta_2,\ldots,\zeta_N)\in\bar{D}(f)} |f(\zeta_1,\zeta_2,\ldots,\zeta_N)|.$

On the other hand, since

$$\|f(T_1,T_2,\ldots,T_N)\| \ge |f(\lambda_1,\lambda_2,\ldots,\lambda_N)|$$

for any

$$(\lambda_1,\ldots\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)$$

$$\prod_{k=1}^{N} \sigma(T_k) \subseteq \bar{D}(f)$$

and $f(\zeta_1,\zeta_2,\ldots,\zeta_N)$ is holomorphic, therefore $|f(\zeta_1,\zeta_2,\ldots,\zeta_N)|$ is holomorphic. Thus there exists a point $(\zeta_1^{(0)},\zeta_2^{(0)},\ldots,\zeta_N^{(0)})$ in $\bar{D}(f)$ such that

$$|f(\zeta_1^{(0)},\zeta_2^{(0)},\ldots,\zeta_N^{(0)})| = \|f(T_1,T_2,\ldots,T_N)\|.$$

Corollary 1. If $T_k\in B(H,H)$ and $T_k = T_k^*$ $(k = 1,2,\ldots,N)$, then there exists a point $(\lambda_1^{(0)},\lambda_2^{(0)},\ldots,\lambda_N^{(0)})$ in $\prod_{k=1}^{N} [m_k,M_k]$ such that

$$|f(\lambda_1^{(0)},\lambda_2^{(0)},\ldots,\lambda_N^{(0)})| = \|f(T\ ,T\ ,\ldots,T_N)\|.$$

The proof is immediate.

If one of the $\zeta_k^{(0)}$ $(1\le k\le N)$ is in $\rho(T_k)$, then $(\zeta_1^{(0)},\ldots,\zeta_2^{(0)})$ $\bar{\in}$ $\prod_{k=1}^{N} \sigma(T_k)$, whence

$$|f(\zeta_1^{(0)},\ldots,\zeta_N^{(0)})| \not> \|f(T_1,T_2,\ldots,T_N)\| .$$

Therefore $\zeta_k^{(0)} \in \sigma(T_k)$ for $k = 1,2,\ldots,N$, that is

$$(\zeta_1^{(0)},\zeta_2^{(0)},\ldots,\zeta_N^{(0)}) \in \prod_{k=1}^{N} \sigma(T_k)$$

in the proof of Theorem 9. But, in general

$$|f(\zeta_1,\zeta_2,\ldots,\zeta_N)| \leq \|f(T_1,T_2,\ldots,T_N)\|$$

for

$$(\zeta_1,\zeta_2,\ldots,\zeta_N) \in \prod_{k=1}^{N} \sigma(T_k).$$

Hence we have the following:

Corollary 2. For the norm of $f(T_1,T_2,\ldots,T_N) \in A[T_1,T_2,\ldots,T_N]$, we have

$$\|f(T_1,T_2,\ldots,T_N)\| = \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N) \in \prod_{k=1}^{N} \sigma(T_k)} |f(\lambda_1,\lambda_2,\ldots,\lambda_N)| . \quad (3\text{-}4)$$

Since $\prod_{k=1}^{N} \sigma(T_k)$ is closed, supremum can be replaced by maximum, that is,

$$\|f(T_1,T_2,\ldots,T_N)\| = \operatorname{Max}_{(\lambda_1,\lambda_2,\ldots,\lambda_N) \in \prod_{k=1}^{N} \sigma(T_k)} |f(\lambda_1,\lambda_2,\ldots,\lambda_N)| . \quad (3\text{-}4')$$

By using Corollary 2, we shall prove the following.

Theorem 10. The set $A[T_1,T_2,\ldots,T_N]$ is a Banach algebra.

Proof. We know that this set is an algebra, and we would like to prove that it is a Banach space. Since

$$\|f(T_1,T_2,\ldots,T_N)+g(T_1,T_2,\ldots,T_N)\|$$

$$= \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N) \in \prod_{k=1}^{N} \sigma(T_k)} |f(\lambda_1,\lambda_2,\ldots,\lambda_N)+g(\lambda_1,\lambda_2,\ldots,\lambda_N)|$$

$$\leq \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)} |f(\lambda_1,\lambda_2,\ldots,\lambda_N)|$$

$$+ \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)} |g(\lambda_1,\lambda_2,\ldots,\lambda_N)|$$

$$= \|f(T_1,T_2,\ldots,T_N)\| + \|g(T_1,T_2,\ldots,T_N)\|$$

and the other axioms of norm are trivial, $A[T_1,T_2,\ldots,T_N]$ is a normed linear space.

The completeness is proved in the following way:
Let $\{f_n(T_1,T_2,\ldots,T_N)\}\subset A[T_1,T_2,\ldots,T_N]$ be a Cauchy sequence, that is, for any $\varepsilon > 0$

$$\|f_n(T_1,T_2,\ldots,T_N)-f_m(T_1,T_2,\ldots T_N)\| < \varepsilon, \quad \text{for} \quad m,n \geq M.$$

Hence

$$\sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)} |f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)-f_m(\lambda_1,\lambda_2,\ldots,\lambda_N| < \varepsilon$$

$$\text{for} \quad m,n \geq M,$$

$$\therefore \ |f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)-f_m(\lambda_1,\lambda_2,\ldots,\lambda_N)| < \varepsilon.$$

Thus the sequence $\{f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)\}$ is Cauchy in $\mathbb{C}$. Therefore there exists a holomorphic function $f_0(\lambda_1,\lambda_2,\ldots,\lambda_N)$ such that

$$\sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N)\in\prod_{k=1}^{N}\sigma(T_k)} |f_n(\lambda_1,\lambda_2,\ldots,\lambda_N)-f_0(\lambda_1,\lambda_2,\ldots,\lambda_N)| \leq \varepsilon$$

and

$$f_0(\lambda_1,\lambda_2,\ldots,\lambda_N) \in H[\bar{D}].^*$$

Hence, according to Lemma 2, this corresponds to an operator

$$f_0(T_1,T_2,\ldots,T_N) \in A[T_1,T_2,\ldots,T_N].$$

Finally,

$$\|f(T_1,T_2,\ldots,T_N)g(T_1,T_2,\ldots,T_N)\|$$

$$= \sup_{(\lambda_2,\lambda_1,\ldots,\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)} \{|f(\lambda_1,\lambda_2,\ldots,\lambda_N)\cdot g(\lambda_1,\lambda_2,\ldots,\lambda_N)|\}$$

$$\leq \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)} |f(\lambda_1,\lambda_2,\ldots,\lambda_N)|$$

$$\times \sup_{(\lambda_1,\lambda_2,\ldots,\lambda_N)\in \prod_{k=1}^{N} \sigma(T_k)} |g(\lambda_1,\lambda_2,\ldots,\lambda_N)|$$

$$= \|f(T_1,T_2,\ldots,T_N)\|\cdot\|g(T_1,T_2,\ldots,T_N)\|$$

i.e., $\|f(T_1,T_2,\ldots,T_N)\cdot g(T_1,T_2,\ldots,T_N)\|$

$$\leq \|f(T_1,T_2,\ldots,T_N)\|\cdot\|g(T_1,T_2,\ldots,T_N)\|.$$

We have proved all required conditions for a Banach algebra.

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* See Hille's Analytic Function Theory I, p.74.

CHAPTER IV

SPECTRUM FOR AN OPERATOR $f(T_1 \otimes T_2)$

1. Some Properties for the Spectral Set $\sigma(T_1 \otimes T_2)$ of the Tensor Product of Operators.

Up to this point, we have discussed spectra on the generalized Dunford integral. Now we turn our attention to another generalized integral:

$$f(T_1 \otimes T_2) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - T_1 \otimes T_2}\, d\zeta,$$

where $T_1 \otimes T_2$ is the tensor product of two operators which belong to $B(H,H)$. For the spectrum of $f(T_1 \otimes T_2)$, we have

$$f[\sigma(T_1 \otimes T_2)] \subseteq \sigma[f(T_1 \otimes T_2)].^*$$

In order to prove this, we shall discuss some preliminary results and definitions. For the tensor product of two operators $T_1, T_2 \in B(H,H)$ we know that

$$\sigma(T_1 \otimes T_2) = \sigma(T_1)\cdot\sigma(T_2).^\dagger$$

Thus

$$(\sigma(T_1 \otimes T_2))^n = \sigma(T_1)^n \cdot \sigma(T_2)^n. \tag{1-1}$$

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* This will be proved in Theorem 11.

† See Reference 3.

On the other hand, we have the following:

Lemma 3. For any positive integer n we have

$$\sigma(T_1^n \otimes T_2^n) = \sigma(T_1)^n \cdot \sigma(T_2)^n. \tag{1-2}$$

Proof. It is well known that

$$\sigma(T^n) = \sigma(T)^n \qquad \text{for} \qquad T \in B(H,H).^{\dagger}$$

For any

$$\mu \in \sigma(T_1)^n \cdot \sigma(T_2)^n = \sigma(T_1^n) \cdot \sigma(T_2^n),$$

$$\mu = \lambda_1^n \lambda_2^n = (\lambda_1 \lambda_2)^n, \qquad \lambda_i \in \sigma(T_i) \quad (i = 1,2),$$

$$(\lambda_1 \cdot \lambda_2)^n - T_1^n \otimes T_2^n = (\lambda_1 \lambda_2 - T_1 \otimes T_2)[(\lambda_1 \lambda_2)^{n-1} + \ldots + (T_1 \otimes T_2)^{n-1}].$$

Thus

$$(\lambda_1 \lambda_2)^n \in \sigma(T_1^n \otimes T_2^n) \qquad \text{since} \qquad \lambda_1 \lambda_2 \in \sigma(T_1 \otimes T_2),$$

$$\text{i.e.,} \quad \sigma(T_1)^n \cdot \sigma(T_2)^n \subseteq \sigma(T_1^n \otimes T_2^n).$$

The opposite inclusion is proved in the following way: Let n = 2.

$$T_1^2 \otimes T_2^2 = (T_1 \otimes T_2)(T_1 \otimes T_2) = (T_1 \otimes I)(I \otimes T_2)(T_1 \otimes I)(I \otimes T_2)$$

and commutativity shows that

$$T_1^2 \otimes T_2^2 = (T_1^2 \otimes I)(I \otimes T_2^2).$$

Moreover,

$$\sigma(T_1^2 \otimes I) = \sigma(T_1^2) \qquad \text{and} \qquad \sigma(I \otimes T_2^2) = \sigma(T_2^2),$$

† See Reference 10.

whence we have

$$(T_1^2 \otimes T_2^2) \subseteq \sigma(T_1^2 \otimes I)\cdot\sigma(I \otimes T_2^2)$$

$$= \sigma(T_1^2)\cdot\sigma(T_2^2)$$

$$= \sigma(T_2)^2\cdot\sigma(T_2)^2.$$

By induction, we get

$$\sigma(T_1^n \otimes T_2^n) \subseteq \sigma(T_1)^n\cdot\sigma(T_2)^n.$$

Therefore

$$\sigma(T_1^n \otimes T_2^n) = \sigma(T_1)^n\cdot\sigma(T_2)^n.$$

The two relations (1-1) and (1-2) give the following equality,

$$[\sigma(T_1 \otimes T_2)]^n = \sigma(T_1^n \otimes T_2^n) = \sigma[(T_1 \otimes T_2)^n]. \tag{1-3}$$

Lemma 4.

$$\sigma[\alpha(T_1 \otimes T_2)^n] = \alpha\cdot\sigma[(T_1 \otimes T_2)^n] \qquad \text{for} \qquad \alpha\in\mathbb{C}-\{0\}.$$

Proof. First of all, we show that

$$\alpha\sigma(T) = \sigma(\alpha T).$$

Since

$$\alpha\lambda-\alpha T = \alpha(\lambda-T) \qquad \text{for any} \qquad \lambda\in\sigma(T),$$

we have

$$\alpha\lambda\in\sigma(\alpha T).$$

Conversely, for an arbitrary $\mu\in\sigma(\alpha T)$, suppose that

$$\mu\notin\alpha\sigma(T),$$

then

$$\frac{\mu}{\alpha} \bar{\in} \sigma(T).$$

Therefore there exists an inverse operator

$$\left(\frac{\mu}{\alpha} - T\right)^{-1} = \left[\frac{1}{\alpha}(\mu - \alpha T)\right]^{-1} = \alpha(\mu - \alpha T)^{-1}$$

$$\therefore \quad \mu \bar{\in} \sigma(\alpha T).$$

This contradiction leads us to the conclusion that

$$\sigma(\alpha T) \subseteq \alpha \cdot \sigma(T).$$

Thus we have proved the equality $\sigma(\alpha T) = \alpha\sigma(T)$.

Let β be a n-th root of α, that is,

$$\beta^n = \alpha.$$

Then

$$\sigma[\beta^n (T_1 \otimes T_2)^n] = \sigma[(\beta T_1 \otimes T_2)^n]$$

$$= (\sigma[\beta T_1 \otimes T_2])^n = [\beta\sigma(T_1 \otimes T_2)]^n = \beta^n[\sigma(T_1 \otimes T_2)]^n$$

$$= \alpha \cdot \sigma[(T_1 \otimes T_2)^n].$$

This completes the proof of the Lemma.

Lemma 5.

$$a_n \sigma[(T_1 \otimes T_2)^n] + a_m \sigma[(T_1 \otimes T_2)^m]$$

$$\subseteq \sigma[a_n (T_1 \otimes T_2)^n + a_m (T_1 \otimes T_2)^m]$$

$$\text{for} \quad a_n, a_m \in \mathbb{C} - \{0\}.$$

Proof. For any $\lambda\epsilon\sigma[(T_1 \times T_2)]$, $\lambda = \lambda_1 \cdot \lambda_2$ where λ_1, λ_2 are contained in $\sigma(T_1)$, $\sigma(T_2)$ respectively.

$$\therefore\ a_n\lambda^n + a_m\lambda^m - [a_n(T_1 \otimes T_2)^n + a_m(T_1 \otimes T_2)^m]$$

$$= a_n[\lambda^n - (T_1 \otimes T_2)^n] + a_m[\lambda^m - (T_1 \otimes T_2)^m]$$

$$= (\lambda - T_1 \otimes T_2)[a_n(\lambda^{n-1} + \ldots + (T_1 \otimes T_2)^{n-1}) + a_m(\lambda^{m-1} + \ldots + (T_1 \otimes T_2)^{m-1})]$$

$$\therefore\quad a_n\lambda^n + a_m\lambda^m \epsilon\, \sigma[a_n(T_1 \otimes T_2)^n + a_m(T_1 \otimes T_2)^m],$$

since $\lambda\epsilon\sigma(T_1 \otimes T_2)$. Thus we have the desired inclusion. We note that for any non-zero constant operator $C = C \cdot I$, $\sigma(CI) = C$.

We define the norm of the tensor product $x \otimes y \epsilon H \otimes H$ to be

$$\|x \otimes y\|^2 = (x \otimes y, x \otimes y) = (x,x)(y,y) = \|x\|^2 \cdot \|y\|^2.$$

Since

$$\|x \otimes y\| \geq 0, \quad \|x \otimes y\| = 0 \quad \text{iff} \quad x \otimes y = 0,$$

$$\|\alpha(x \otimes y)\| = |\alpha| \|x \otimes y\|$$

$$\|x_1 \otimes y_1 + x_2 \otimes y_2\|^2 = (x_1 \otimes y_1 + x_2 \otimes y_2, x_1 \otimes y_1 + x_2 \otimes y_2)$$

$$= \|x_1 \otimes y_1\|^2 + \|x_2 \otimes y_2\|^2 + 2\mathrm{Re}(x_1 \otimes y_1, x_2 \otimes y_2)$$

$$\leq \|x_1 \otimes y_1\|^2 + \|x_2 \otimes y_2\|^2 + 2|(x_1 \otimes y_1, x_2 \otimes y_2)|$$

$$\leq \|x_1 \otimes y_1\|^2 + \|x_2 \otimes y_2\|^2 + 2\|x_1 \otimes y_1\| \|x_2 \otimes y_2\|$$

$$= (\|x_1 \otimes y_1\| + \|x_2 \otimes y_2\|)^2$$

$$\therefore \|x_1 \otimes y_1 + x_2 \otimes y_2\| \leq \|x_1 \otimes y_1\| + \|x_2 \otimes y_2\|,$$

thus the norm is well defined. And we define the norm $\|T_1 \otimes T_2\|$ by

$$\|T_1 \otimes T_2\| = \sup_{\|x \otimes y\| \leq 1} \|(T_1 \otimes T_2)(x \otimes y)\| = \sup_{\|x \otimes y\| \leq 1} \|T_1 x \otimes T_2 y\|.$$

This also satisfies three of the axioms of norm: For the triangularity

$$\|(A_1 \otimes B_1 + A_2 \otimes B_2)(x \otimes y)\| = \|(A_1 \otimes B_1)(x \otimes y) + (A_2 \otimes B_2)(x \otimes y)\|$$

$$\leq \|(A_1 \otimes B_1)(x \otimes y)\| + \|(A_2 \otimes B_2)(x \otimes y)\|$$

since

$$Ax \otimes By \in H \otimes H.$$

Hence we have

$$\|A_1 \otimes B_1 + A_2 \otimes B_2\| \leq \|A_1 \otimes B_1\| + \|A_2 \otimes B_2\|.$$

We consider the collection

$$\{T_1 \otimes T_2 \mid T_1 \otimes T_2 : H \otimes H \to H \otimes H\}, \quad T : H \to H,$$

which is defined by

$$(T_1 \otimes T_2)(x \otimes y) = T_1 x \otimes T_2 y.$$

With the above norm, $(\{T_1 \otimes T_2\}, \| \ \|)$ forms a metric space. By definition

$$\|T_1 \otimes T_2\| = \sup_{\|x \otimes y\| \leq 1} \|T_1 x \otimes T_2 y\| \leq \sup_{\|x \otimes y\| \leq 1} \|T_1 x\| \cdot \|T_2 y\|$$

$$\leq \sup_{\|x\| \leq 1} \|T_1 x\| \cdot \sup_{\|y\| \leq 1} \|T_2 y\| = \|T_1\| \cdot \|T_2\|$$

$$\therefore \qquad \|T_1 \otimes T_2\| \le \|T_1\|\cdot\|T_2\|.$$

The spectrum $\sigma[T_1 \otimes T_2] = \sigma(T_1)\cdot\sigma(T_2)$ is closed and bounded since $\sigma(T_1)$ and $\sigma(T_2)$ are closed and bounded in the sense of usual topology in the complex plane, that is,

$$|\zeta| = |\lambda_1\cdot\lambda_2| = |\lambda_1|\cdot|\lambda_2| \le \|T_1\|\cdot\|T_2\| \qquad \text{for any} \qquad \zeta\in\sigma(T_1 \otimes T_2).$$

A sharper inequality is now given:

According to the definition of spectrum, we have

$$|\zeta| \le \|T_1 \otimes T_2\| \qquad \text{if} \qquad \zeta\in\sigma(T_1 \otimes T_2).$$

Thus

$$\sup_{\zeta\in\sigma(T_1 \otimes T_2)} |\zeta| \le \|T_1 \otimes T_2\| \le \|T_1\|\|T_2\|.$$

Therefore $\sigma(T_1 \otimes T_2)$ is bounded, since T_1, T_2 are bounded. Thus $\sigma(T_1 \otimes T_2)$ lies entirely inside of a circle with radius less than $\|T_1 \otimes T_2\|$ and centered at origin in the complex plane.

Now, we return to the discussion of the spectrum. By the same argument as given in Lemma 5, we have

$$\sigma[\sum_{n=0}^{N} a_n(T_1 \otimes T_2)] \supseteq \sum_{n=0}^{N} a_n\sigma[(T_1 \otimes T_2)^n] \qquad (1\text{-}4)$$

for any natural number N. The proof is immediate since $\sigma[a_o I] = a_o$.

We would like to prove the following:

Lemma 6.

$$\sigma[\sum_{n=0}^{\infty} a_n(T_1 \otimes T_2)^n] \supseteq \sum_{n=0}^{\infty} a_n\ [(T_1 \otimes T_2)^n]$$

if both exist.

Proof. To prove this, it is sufficient to prove that

$$\sigma[\lim_{N\to\infty} \sum_{n=0}^{N} a_n (T_1 \otimes T_2)^n] = \lim_{N\to\infty} \{\sigma[\sum_{n=0}^{N} a_n (T_1 \otimes T_2)^n]\}.$$

But this is equivalent to the continuity of the function σ with respect to the operator $\sum_{n=0}^{N} a_n (T_1 \otimes T_2)^n$, that is,

$$\sigma[\sum_{n=0}^{N} a_n (T_1 \otimes T_2)^n] \to \sigma[\sum_{n=0}^{\infty} a_n (T_1 \otimes T_2)^n] \quad \text{as} \quad N \to \infty.$$

The proof is obvious. Thus

$$\sigma[\lim_{N\to\infty} \sum_{n=0}^{N} a_n (T_1 \otimes T_2)^n] = \lim_{N\to\infty} \sigma[\sum_{n=0}^{N} a_n (T_1 \otimes T_2)^n]$$

$$\supseteq \lim_{N\to\infty} \sum_{n=0}^{N} a_n \cdot \sigma(T_1 \otimes T_2)^n$$

$$\text{i.e.,} \quad [\sum_{n=0}^{\infty} a_n (T_1 \otimes T_2)^n] \supseteq \sum_{n=0}^{\infty} a_n \cdot \sigma(T_1 \otimes T_2)^n.$$

2. The Spectral Set $\sigma[f(T_1 \otimes T_2)]$.

Now we are prepared to prove the following result.

Theorem 11. Defining $f(T_1 \otimes T_2) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - T_1 \otimes T_2}\, d\zeta$ we have the relation

$$f(\sigma(T_1 \otimes T_2)) \subseteq \sigma[f(T_1 \otimes T_2)]$$

where $f(\zeta)$ is holomorphic on $\bar{D}(f) \supseteq U \cup C$, $U \supset \sigma(T_1 \otimes T_2)$.

Proof. Since

$$[\zeta - (T_1 \otimes T_2)]^{-1} = \sum_{n=0}^{\infty} \zeta^{-(n+1)} (T_1 \otimes T_2)^n$$

is uniformly convergent on the s.c.o.r.c. C, we get

$$f(T_1 \otimes T_2) = \frac{1}{2\Pi i}\int_C f(\zeta) \sum_{n=0}^{\infty} \zeta^{-(n+1)} (T_1 \otimes T_2)^n d\zeta$$

$$= \sum_{n=0}^{\infty} (T_1 \otimes T_2)^n \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-0)^{n+1}}\, d\zeta$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (T_1 \otimes T_2)^n.$$

Therefore,

$$\sigma[f(T_1 \otimes T_2)] = \sigma[\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (T_1 \otimes T_2)^n]$$

$$\supseteq \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} [\sigma(T_1 \otimes T_2)]^n = \{\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \lambda^n : \lambda \epsilon \sigma(T_1 \otimes T_2)\}$$

$$= \{f(\lambda) : \lambda \epsilon \sigma(T_1 \otimes T_2)\} = f(\sigma(T_1 \otimes T_2)).$$

We have proved the theorem.

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Jae Chul Rho was born January 17, 1929, in Sang-Ju, South Korea. In March, 1949, he was graduated from Junior College (High School), Teacher's College, Taegu, Korea. In March, 1953, he received the degree of Bachelor of Science from Teacher's College in Kyung Pook University, and in March, 1958, he received the degree of Master of Science from the Graduate School, Kyoung Pook University. From 1954, until 1956, he taught Mathematics at the Attached High School, Kyoung Pook University. From 1958, until 1960, he taught Mathematics at Chung-Ju College as an Instructor, and he also taught Mathematics at the Dongkook University, Seoul, Korea from 1960, until July, 1963, as an Assistant Professor.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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